CONTEMPORARY MATHEMATICS

191

Representation Theory and Harmonic Analysis

A Conference in Honor of Ray A. Kunze
January 12–14, 1994
Cincinnati, Ohio

Tuong Ton-That, Coordinating Editor

Kenneth I. Gross
Donald St. P. Richards
Paul J. Sally, Jr.
Editors

American Mathematical Society

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Ray Alden Kunze

Representation Theory and Harmonic Analysis

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Providence, Rhode Island

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Dedication

This volume is dedicated to Ray A. Kunze in recognition of his long and distinguished career as a leading researcher in group representations and harmonic analysis.

Ray Alden Kunze was born March 7, 1928, in Des Moines, Iowa, but lived much of his youth in the area around Milwaukee, Wisconsin. He attended Denison University for two years before transferring to the University of Chicago, where he received his Bachelor of Science and Master of Science degrees, both in mathematics. Ray's graduate studies were interrupted during the early 1950's by military service, during which period he served as a mathematical analyst in the Department of Defense. Upon completion of his tour of duty, Ray returned to his doctoral studies at the University of Chicago, and in 1957 received his Doctor of Philosophy degree in mathematics. His dissertation was written under the supervision of Irving E. Segal.

Ray then went to the Massachusetts Institute of Technology as a C.L.E. Moore Instructor, and over the ensuing years has served on the faculties of Brandeis University, Washington University, the University of California at Irvine, and the University of Georgia. At both Irvine and Georgia he chaired the Department of Mathematics. Ray has also held numerous visiting positions in the United States and abroad.

Ray's mathematical research is characterized by exquisite mathematical taste, deep creative insight, algebraic elegance, and analytical power. His numerous publications include seminal contributions to almost every aspect of group representations and noncommutative harmonic analysis: Fourier transforms on locally compact groups, noncommutative interpolation theory, intertwining operators for the principal series and uniformly bounded representations of semisimple Lie groups, operator-valued reproducing kernels, holomorphic multiplier representations, generalized Bessel functions and Hankel transforms, analysis on matrix space, the holomorphic discrete series and related representations, the metaplectic (or Segal-Shale-Weil, or oscillator) representation, algebraic induction, invariant theory, and exceptional representations.

Ray is also well-known for his clarity of mathematical exposition. His celebrated textbook on linear algebra, coauthored with Kenneth M. Hoffman, has

been a classic for over three decades and has been translated into several languages. As further testimonial to his reputation as a lecturer, Ray frequently has been invited to teach graduate level summer school courses, for example at NSF and NATO institutes and at universities in Italy, France, and Australia.

No summary of Ray's professional contributions would be complete without taking special note of his extraordinary talents as a mentor of doctoral students. Those who have had the pleasure and privilege of being his student have had the best mathematical training possible. His immense patience in guiding a student from course work to independent research, his clear and precise explanations, his unerring ability to direct a student toward mainstream problems of importance on which one could make steady progress, his unwavering support and encouragement when the going gets rough, his network of contacts through which he helps a student find a position upon completion of the degree, and the lifelong mutual affection and friendship that characterizes his relationship with his students, make Ray a perfect model of what a thesis advisor can and should be.

With pride and great pleasure, we dedicate this volume to Ray as a tribute to his distinguished career as a research mathematician and inspirational teacher. We wish him many more years of good health, happiness, and continued deep and important contributions to mathematics.

Preface

This volume is an outgrowth of the special session on representation theory and harmonic analysis held in honor of Ray Kunze at the 889th meeting of the American Mathematical Society on January 12-15, 1994, at Cincinnati, Ohio. The mathematicians who gathered together for the special session included many of Ray's former Ph.D. students, research collaborators, and others whose work has been influenced either directly or indirectly through their associations with Ray. As the session progressed, it became clear that the papers being presented spanned a wide spectrum of group representations and applications that would be of interest to a more extensive audience.

For the most part, the papers that appear here are based upon lectures presented at the special session.[1] Although the central theme is group representations and its applications to harmonic analysis, this volume contains a diversity of mathematics. A glance through the table of contents, for example, reveals that the algebraic and geometric points of view in representation theory are also greatly in evidence, and that quite classical analysis in higher dimensions, not particularly closely tied to representation theoretic ideas, appears as well. To convey both the richness and extent of the mathematics included in the volume, we note that the papers relate to many topics, such as the following: analysis, algebra, and geometry on Riemannian, pseudo-Riemannian, and locally Riemannian symmetric spaces; Lie algebra cohomology; projective and integral geometry; representations of semi-simple Lie groups, including the holomorphic discrete series and highest weight representations more generally, non-holomorphic discrete series, the principal series, and exceptional representations; invariant theory and the oscillator representation; invariant differential operators and the work of Harish-Chandra; p-adic representation theory; and classical Fourier analysis.

In closing, the editors are pleased to recognize and thank two other individuals who helped produce this volume: Barbara Palmore, whose excellent typing and

[1] A few of the lectures treated research that had already been submitted elsewhere, and of course could not be included in this volume. However, in some cases the topic does appear in the volume as an expository overview of the subject that is not available elsewhere. Conversely, some invited speakers who were unable to attend the special session submitted their work in written form to these proceedings.

formatting of the manuscripts is reflected in the finished product, and Donna Harmon of the A.M.S. staff who provided expert production advice. We also wish to thank the referees for their careful reading of the papers and their helpful comments to authors. Most importantly, the editors are indebted to the speakers at the special session for helping make the event so memorable, and for their encouragement to produce this volume.

Tuong Ton-That
Kenneth I. Gross
Donald St. P. Richards
Paul J. Sally, Jr.

Contemporary Mathematics
Volume 191, 1995

Supercuspidal character formulas for GL_ℓ

Lawrence Corwin†, Allen Moy, and Paul J. Sally, Jr.

0. Introduction

The irreducible unitary representations of the multiplicative group $D_n^\times$ of a division algebra of degree n (dimension n^2) over a p-adic field F were determined by Corwin [**1**] and Howe [**7**] in the case when $(n,p)=1$. Corwin and Howe showed that the representations are parametrized by admissible characters of extensions E/F of degree m where $m|n$. In a subsequent paper [**2**], Corwin and Howe began a study of the characters of these representations. Any extension E/F of the above type can be embedded in both $D_n^\times$ and $\mathrm{GL}_n(F)$, and the matching theorem states that the irreducible characters of $D_n^\times$ on $E^\times$ are equal, up to a sign, to the discrete series characters of $\mathrm{GL}_n(F)$ on $E^\times$ regarded as an elliptic Cartan subgroup in $\mathrm{GL}_n(F)$. Thus, the discrete series characters for $\mathrm{GL}_n(F)$ on the elliptic set can be obtained from the irreducible characters of $D_n^\times$. The full details will be presented in a paper by the authors [**3**].

There is little quantitative information available on the discrete series characters for reductive p-adic groups. The characters for $\mathrm{SL}_2(F)$ [**12**], $\mathrm{PGL}_2(F)$ [**14**], and $\mathrm{GL}_2(F)$ [**13**] have been known for some time in the case $p \neq 2$. Kutzko [**9**] determined the supercuspidal characters of $\mathrm{GL}_\ell(F)$, ℓ prime, without any restriction on the characteristic of F. Kutzko's results take a form which is different from those in the present paper since the characters are not readily expressible as locally constant functions on tori when $p=\ell$. We also note the character formulas in Gérardin [**5**] which hold in a limited range for unramified supercuspidal representations of $\mathrm{GL}_n(F)$ when $(n,p)=1$. The reader should consult the paper [**4**] of the authors for more background and details, and the paper [**11**] for some additional comments. A different perspective on characters, following the work of Bernstein, Kazhdan, and Lusztig may be found in the paper [**6**] of Hales.

In this note, we illustrate the procedure used in the computation of supercuspidal characters for GL_n. In particular, we consider the case in which $n=\ell$, a prime, for which the complications are relatively minor. Assume that D is a division algebra of index ℓ over F. Let $\mathbb{F}$ be the residue class field of F, $\mathbb{D}$ the residue class field of D, $\mathcal{O}$ the integers in D, and P the prime ideal in $\mathcal{O}$. Let $[E:F]=\ell$,

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†Deceased.

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with E totally ramified over F. We start with an admissible character ψ on $E^\times$ and construct $\pi = \pi_\psi \in (D^\times)\hat{}$. We shall then compute the character Θ_π of π.

We embed E in D as follows: there is a prime $\varpi \in D$ with $\varpi^\ell \in F$, and there is a set $\mathbb{D}^\sim$ of representatives for $\mathbb{D} = \mathcal{O}/P$ such that the following statements are true:

(a) $0 \in \mathbb{D}^\sim$ and $\mathbb{D}^\sim \setminus \{0\}$ is a cyclic group;
(b) The map $\alpha \longmapsto \varpi\alpha\varpi^{-1}$ is bijective on $\mathbb{D}^\sim$;
(c) If $\alpha \longmapsto \overline{\alpha}$ is the restriction of the map $\mathcal{O} \longrightarrow \mathcal{O}/P = \mathbb{D}$ to $\mathbb{D}^\sim$, then $\overline{\alpha} \longmapsto \overline{\varpi\alpha\varpi^{-1}}$ generates $\mathrm{Gal}(\mathbb{D}/\mathbb{F})$;
(d) $F[\varpi] \cong E$.

(See, e.g., [**15**].) We define $\sigma \in \mathrm{Gal}(\mathbb{D}/\mathbb{F})$ by $\overline{\alpha}^\sigma = \overline{(\varpi\alpha\varpi^{-1})}$. In fact, σ determines the algebra D.

We shall often regard E as $F[\varpi]$ and write α for $\overline{\alpha}$, as a matter of typographical convenience. However, we shall be careful not to confuse α and $\overline{\alpha}$ in calculations.

The authors would like to express their thanks to Jeff Adler for his helpful and incisive comments.

1. Construction of the representations and the inducing characters

Let ν be the usual valuation on D (with $\nu(\varpi) = 1$). Recall that every element $x \in D$ has a unique expansion in the form

$$x = \sum_{j=j_0}^{\infty} \alpha_j \varpi^j, \qquad \alpha_j \in \mathbb{D}^\sim \text{ and } j_0 = \nu(x) \text{ (hence } \alpha_{j_0} \neq 0).$$

Let $\mathbb{F}^\sim = F \cap \mathbb{D}^\sim$. Then it is easy to see that

$$x \in E \Longleftrightarrow \alpha_j \in \mathbb{F}^\sim \quad \text{for all } j.$$

Let $q = \operatorname{card} \mathbb{F}^\sim = \operatorname{card} \mathbb{F}$.

Note that $\mathrm{Tr}_{\mathbb{D}/\mathbb{F}} \colon \mathbb{D} \longrightarrow \mathbb{F}$ is faithful on $\mathbb{F}$ (in fact, it's just multiplication by ℓ there.) Let $T = \left\{ \gamma \in \mathbb{D} \,\middle|\, \mathrm{Tr}_{\mathbb{D}/\mathbb{F}}\, \gamma = 0 \right\}$, and let $T^\sim$ be the preimage of T in $\mathbb{D}^\sim$. Then $T \oplus \mathbb{F} = \mathbb{D}$.

We now construct π. Recall from [**8**] that (in our present, prime-dimensional, totally ramified case) a character ψ of $E^\times$ is *admissible* over F if and only if $\psi|_{1+P_E}$ does not factor through the norm from E to F. Any admissible character has a Howe factorization

$$\psi = \psi_1(\psi_2 \circ \mathrm{N}_{E/F}),$$

where ψ_2 is chosen to minimize the conductor of ψ_1 (see, for example, [**10**]). For simplicity, we shall assume that ψ already has minimal conductor, so that $\psi = \psi_1$ and twisting by ψ_2 is unnecessary. (We observe that twisting by ψ_2 only changes the central character of the resulting representation of $D^\times$.) This means that the conductor $m + 1$ of ψ satisfies

$$(m, \ell) = 1.$$

Now consider $\psi|_{(1+P^m)\cap E}$. By the definition of m, $\psi|_{(1+P^{m+1})\cap E} \equiv 1$. Therefore, $\psi|_{1+P^m}$ is determined by $\psi(1 + \alpha\varpi^m)$, $\alpha \in \mathbb{F}^\sim$. Thus there is an additive character ψ_0 of $\mathbb{F}$ such that

$$\psi(1 + \alpha\varpi^m) = \psi_0(\mathrm{Tr}\,\alpha) = \psi_0(\ell\alpha);$$

Moreover, ψ_0 is nontrivial.

Now define

$$\begin{cases} m' = m'' = \frac{m+1}{2} & \text{if } m \text{ is odd;} \\ m' = \frac{m}{2} + 1, \ m'' = \frac{m}{2} & \text{if } m \text{ is even;} \end{cases}$$

let $H' = E^\times(1+P^{m'})$, $H'' = E^\times(1+P^{m''})$, and $H = E^\times(1+P)$. We want to extend ψ to $\psi^\sharp$ on H'. For this purpose, we set

$$K_0 = \left\{ y \in P^{m'} \;\middle|\; \mathrm{Tr}_{D/F}\, yz = 0 \text{ for all } z \in E \right\},$$

and define

$$K = (1+P^{m+1}) \cdot (1+K_0).$$

Then K is a normal subgroup of H', and $K \cap E^\times \subset 1+P^{m+1}$. We observe that $H' = K \cdot E^\times$, and we extend ψ by setting

$$\psi^\sharp|_K \equiv 1, \quad \psi^\sharp|_{E^\times} = \psi.$$

Note that $\psi^\sharp(1+\beta\varpi^m) = \psi_0 \circ \mathrm{Tr}_{\mathbb{D}/\mathbb{F}}(\beta)$ for *all* $\beta \in \mathbb{D}^\sim$, since $\mathbb{D} = \mathbb{F} \oplus T$. It will be useful later to know that if $y \in K_0$, with $y = \sum_{j=j_0}^\infty \gamma_j \varpi^j$ (here, $j_0 \geq m'$), then $\gamma_{j_0} \in T^\sim$.

Next, we extend $\psi^\sharp$ to $\psi^\sim$ on H''. When m is odd, this is trivial, since $H' = H''$. When m is even, the extension can be done via a Weil representation (as in [**10**]). In this case, we need the following facts about $\psi^\sim$:

1. $\dim \psi^\sim = q^{\frac{\ell-1}{2}}$ (recall $q = \mathrm{card}\,\mathbb{F}$; $q^{\ell-1} = \mathrm{card}\,T$).
2. On $F^\times \cdot (1+P) \cap H'$, $\psi^\sim$ is a multiple of $\psi^\sharp$.
3. On elements of E of the form $x = \alpha_0 \varpi^{j_0}(1 + \alpha_1 \varpi + \cdots)$, with $\alpha_0 \varpi^{j_0} \notin F$, we have (when q is odd)

$$\psi^\sim(x) = \lambda(E/F)\psi(x),$$

where, for E/F ramified, $\lambda(E/F) = 1$ if q is a square mod ℓ and -1 otherwise. (See [**10**].)

Define $\rho = \mathrm{Ind}_{H''}^H \psi^\sim$, $\pi = \mathrm{Ind}_H^{D^\times} \rho$. We want to compute Θ_π. There are three main stages in this procedure:

(i) computing $\theta_{\psi^\sim}$ from ψ,
(ii) computing θ_ρ from $\theta_{\psi^\sim}$, and
(iii) computing Θ_π from θ_ρ.

The first stage (or, at least, as much as we shall need of $\theta_{\psi^\sim}$) amounts to the three facts about $\psi^\sim$ listed above.

The second stage is the longest and hardest. We break it into steps.

2. Restricting the support of θ_ρ

We state a lemma which will be proved in full generality in [**3**]. Here we will be content to prove a special case, which gives the flavor of the general proof.

We say that $x = \sum_{j=j_0}^\infty \alpha_j \varpi^j$ ($\alpha_j \in \mathbb{D}^\sim$ for all j) is *normal* if the terms $\alpha_j \varpi^j$ all commute.

LEMMA 2.1. *Let D be a division algebra of index n over F, with $(n,p) = 1$. Given $y \in D$, there is a unique normal $x \in D$ conjugate to y under $1+P$.*

PROOF IN A SPECIAL CASE: Assume that $y = \sum_{j=1}^{\infty} \alpha_j \varpi^j$, with $\alpha_1 = 1$. We show that y is conjugate under $1+P$ to a unique $x \in D$ with

$$x = \sum_{j=1}^{\infty} \beta_j \varpi^j, \qquad \text{all } \beta_j \in \mathbb{F}^{\sim}.$$

The proof is inductive. Conjugate y with an arbitrary element of $1+P$, and compute mod P^3, remembering that $\varpi\gamma = \gamma^\sigma \varpi$ for $\gamma \in \mathbb{D}^{\sim}$:

$$\begin{aligned}
&(1+\delta\varpi+\delta'\varpi^2+\cdots)y(1+\delta\varpi+\delta'\varpi^2+\cdots)^{-1} \\
&\quad = (1+\delta\varpi+\cdots)y(1-(\delta\varpi+\cdots)+(\delta\varpi+\cdots)^2) \\
&\quad \equiv y + (\delta\alpha_1^\sigma - \alpha_1\delta^\sigma)\varpi^2 \pmod{P^3} \\
&\quad \equiv \varpi + (\alpha_2 + \delta - \delta^\sigma)\varpi^2 \pmod{P^3},
\end{aligned}$$

since $\alpha_1 = 1$. Now in $\mathbb{D}$, $\delta - \delta^\sigma$ can be any element of T; hence $\alpha_2 + \delta - \delta^\sigma$ can be any element of $\mathbb{D}$ with trace equal to that of α_2. Hence there is a unique choice of $\beta_2 = \alpha_2 + \delta - \delta^\sigma$ that is in $\mathbb{F}$ (though δ is not unique). With an appropriate choice of δ, then, y is conjugate to $y_2 = \varpi + \beta_2\varpi^2 + \alpha_3'\varpi^3 + \cdots$, with $\beta_2 \in \mathbb{F}$ (and β_2 uniquely determined). Now repeat, conjugating y_2 by $(1+\delta'\varpi^2+\cdots)$ to produce

$$y_3 = \varpi + \beta_2\varpi^2 + \beta_3\varpi^3 + \alpha_4''\varpi^4 + \cdots,$$

with β_2 and β_3 in $\mathbb{F}$ and uniquely determined. Finally, take the limit of the y_j as $j \to \infty$. □

It follows that we can restrict attention to normal elements x. Write x in the form $x = \alpha_0\varpi^{j_0}(1+\alpha_1\varpi+\alpha_2\varpi^2+\cdots)$. When we write $\alpha_j\varpi^j$, it is understood that this means $\alpha_0\varpi^{j_0}$ when $j = 0$. (Note that this is at variance with the previous use of indices.)

We now reimpose the restriction that $n = \ell$, a prime.

LEMMA 2.2. *Suppose x is a normal element and $x \notin F$. If j is any index such that $\alpha_j\varpi^j \notin F$, then $F[x] = F[\alpha_j\varpi^j]$.*

PROOF. $[F[\alpha_j\varpi^j] : F] = \ell$, since all extensions of F in D have degree ℓ (except F itself). Hence $F[\alpha_j\varpi^j]$ is its own centralizer in D. It follows that every $\alpha_i\varpi^i$ is in $F[\alpha_j\varpi^j]$, and hence $x \in F[\alpha_j\varpi^j]$. Since $[F[x] : F] = \ell$, the conclusion follows. □

LEMMA 2.3. *$\rho|_{1+P^m}$ is a multiple of ψ.*

PROOF. Since $\psi^{\sim}|_{1+P^m}$ is a multiple of ψ, it suffices to show that for some set S of coset representatives for H/H'',

$$w \in S \Longrightarrow \psi(wyw^{-1}) = \psi(y), \qquad \text{all } y \in 1+P^m. \tag{2.4}$$

Since $H/H'' \cong (H \cap (1+P))/(H'' \cap (1+P))$, we can pick the w to be in $1+P$. But $1+P^{m+1}$ is normal in $1+P$, and is contained in the kernel of ψ. So $(1+P^m)/(1+P^{m+1})$ is central in $(1+P)/(1+P^{m+1})$, and (2.4) is obvious. □

PROPOSITION 2.5. *If x is normal and $\theta_\rho(x) \neq 0$, then $x \in E^\times(1+P^m)$.*

PROOF. Note first that the hypothesis and conclusion are unaffected if x is multiplied by an element of $F^\times$.

Write $x = \alpha_0\varpi^{j_0}(1+\alpha_1\varpi+\cdots)$, and assume that the proposition is false for x. Since $\alpha_0\varpi^{j_0} \in E$ by construction of H, we must have $\alpha_0\varpi^{j_0} \in F$; otherwise, Lemma 2.2 gives $x \in E$. By the note above, we may divide by $\alpha_0\varpi^{j_0}$; that is, we

may assume that $j_0 = 0$ and $\alpha_0 = 1$. Let j be the first index with $\alpha_j \varpi^j \notin F$. Then $j < m$ (otherwise $x \in F(1 + P^m)$), and $\alpha_j \notin \mathbb{F}^\sim$.

Now let $x_\delta = (1 + \delta\varpi^{m-j})x(1 + \delta\varpi^{m-j})^{-1}$, $\delta \in \mathbb{D}^\sim$. We have

$$x_\delta \equiv x + (\delta\alpha_j^{\sigma^{m-j}} - \alpha_j\delta^{\sigma^j})\varpi^m \pmod{P^{m+1}}; \tag{2.6}$$

$$x_\delta \equiv x(1 + (\delta\alpha_j^{\sigma^{m-j}} - \alpha_j\delta^{\sigma^j})\varpi^m) \pmod{1 + P^{m+1}}. \tag{2.7}$$

This sort of computation will be needed often in the future; here are the details in this case. Note first that

$$(1 + \delta\varpi^{m-j})^{-1} = 1 - \delta\varpi^{m-j} + (\delta\varpi^{m-j})^2 - \cdots.$$

Now compute $(1 + \delta\varpi^{m-j})\alpha_i\varpi^i(1 + \delta\varpi^{m-j})^{-1}$ for each i. When $i < j$, $\alpha_i\varpi^i$ is central, and the result is $\alpha_i\varpi^i$. When $i > j$, the expanded product has every term except $\alpha_i\varpi^i$ in P^{m+1}; that is,

$$(1 + \delta\varpi^{m-j})\alpha_i\varpi^i(1 + \delta\varpi^{m-j})^{-1} \equiv \alpha_i\varpi^i \pmod{P^{m+1}} \quad \text{for } i > j.$$

For $i = j$, the only terms in the expansion that are not in P^{m+1} are

$$\alpha_i\varpi^i + \delta\varpi^{m-j}\alpha_i\varpi^i - \alpha_i\varpi^i\delta\varpi^{m-j} = \alpha_i\varpi^i + (\delta\alpha_i^{\sigma^{m-j}} - \alpha_i\delta^{\sigma^j})\varpi^m,$$

because $\varpi\gamma = \gamma^\sigma\varpi$ if $\gamma \in \mathbb{D}^\sim$. Now summing gives (2.6), and (2.7) follows from (2.6) by multiplying out terms.

Since θ_ρ is constant on conjugacy classes, Lemma 2.3 gives

$$\theta_\rho(x) = \theta_\rho(x_\delta) = \theta_\rho(x)\psi(1 + (\delta\alpha_j^{\sigma^{m-j}} - \alpha_j\delta^{\sigma^j})\varpi^m),$$

for all $\delta \in \mathbb{D}^\sim$. By our assumption, $\theta_\rho(x) \neq 0$. Therefore, we have (for all $\delta \in \mathbb{D}^\sim$)

$$\begin{aligned} 1 &= \psi(1 + (\delta\alpha_j^{\sigma^{m-j}} - \alpha_j\delta^{\sigma^j})\varpi^m) \\ &= \psi_0 \circ \mathrm{Tr}_{\mathbb{D}/\mathbb{F}}(\delta\alpha_j^{\sigma^{m-j}} - \alpha_j\delta^{\sigma^j}) \\ &= \psi_0 \circ \mathrm{Tr}_{\mathbb{D}/\mathbb{F}}\,\delta^{\sigma^j}(\alpha_j^{\sigma^m} - \alpha_j), \end{aligned}$$

this last holding because $\mathrm{Tr}_{\mathbb{D}/\mathbb{F}}(\gamma) = \mathrm{Tr}_{\mathbb{D}/\mathbb{F}}(\gamma^\sigma)$ for all $\gamma \in \mathbb{D}$. But ψ_0 is nontrivial and $\mathrm{Tr}_{\mathbb{D}/\mathbb{F}}$ is surjective; thus we must have

$$\overline{\alpha_j}^{\sigma^m} - \overline{\alpha_j} = 0, \qquad \overline{\alpha_j}^{\sigma^m} = \overline{\alpha_j}.$$

But σ^m generates $\mathrm{Gal}(\mathbb{D}/\mathbb{F})$, since $\ell \nmid m$ and ℓ is prime. Therefore $\overline{\alpha_j}$ is $\mathrm{Gal}(\mathbb{D}/\mathbb{F})$-invariant, or $\overline{\alpha_j} \in \mathbb{F}$, or $\alpha_j \in \mathbb{F}^\sim$. This gives the desired contradiction, and the theorem is proved. □

3. Computing $\theta_\rho(x)$

We need to examine $\theta_\rho(x)$ more closely for $x \in E^\times(1 + P^m)$. In view of Lemma 2.2, we can treat $E^\times$ and $1 + P^m$ separately, and we have described $\theta_\rho|_{1+P^m}$ in Lemma 2.3. Thus we need only deal with $\theta_\rho|_{E^\times}$.

Given any normal $x \in D$, write $x = \alpha_0\varpi^{j_0}(1 + \alpha_1\varpi + \cdots)$ and define

$$\nu_0(x) = \begin{cases} 0 & \text{if } \alpha_0\varpi^{j_0} \notin F; \\ j & \text{if } \alpha_0\varpi^{j_0} \in F \text{ and } j \text{ is the smallest index with } \alpha_j\varpi^j \notin F. \end{cases}$$

(If $x \in F$, $\nu_0(x) = \infty$.)

When $\nu_0(x) \geq m$, then $x \in F^\times(1 + P^m)$ and we have already described $\theta_\rho(x)$ (modulo knowing $\dim\rho$). For other $x \in E^\times$, we need to know more about H/H''.

LEMMA 3.1. *For* $\gamma = (\gamma_1, \dots, \gamma_{m''-1}) \in (T^\sim)^{m''-1}$, *let*

$$y_\gamma = (1+\gamma_1\varpi)\cdot\dots\cdot(1+\gamma_{m''-1}\varpi^{m''-1}).$$

The y_γ *form a set of coset representatives for* H/H''.

PROOF. Let $H_j = H \cap (1+P^j)$, $H_j'' = H'' \cap (1+P^j)$. Then $H_j = H_j''$ for $j \geq m''$. We noted in the proof of Lemma 2.3 that $H/H'' = H_1/H_1''$. We thus get a complete set of coset representatives for H/H'' by getting coset representatives for $H_j/H_{j+1}H_j''$, $1 \leq j \leq m''-1$, and multiplying them together. Since the elements $1+\gamma_j\varpi^j$, $\gamma_j \in T^\sim$, give such a set of coset representatives, the lemma follows. □

COROLLARY. $\dim \rho = q^{(\ell-1)(m''-1)} \dim \psi^\sim$.

PROOF. $\operatorname{card} T^\sim = q^{\ell-1}$. □

PROPOSITION 3.2. *If* $\nu_0(x) = 0$ *(and* $x \in E$*), then* $\theta_\rho(x) = \theta_{\psi^\sim}(x)$.

PROOF. The standard Frobenius formula gives

$$\theta_\rho(x) = \sum_{\gamma \in T^{\sim m''-1}} \dot\theta_{\psi^\sim}(y_\gamma x y_\gamma^{-1}),$$

where $\dot\theta_{\psi^\sim} = \theta_{\psi^\sim}$ on H'' and 0 elsewhere. It suffices to prove that if $\gamma \neq (0,\dots,0) = 0$, then $y_\gamma x y_\gamma^{-1} \notin H''$.

If we write $x = \alpha_0\varpi^{j_0}(1+\alpha_1\varpi+\cdots)$, then $\ell \nmid j_0$. If $\gamma \neq 0$, let γ_s be the first nonzero coordinate in γ. Then we get

$$y_\gamma x y_\gamma^{-1} \equiv x + (\gamma_s\alpha_0^{\sigma^s} - \alpha_0\gamma_s^{\sigma^{j_0}})\varpi^{j_0+s} \pmod{P^{j_0+s+1}}.$$

Let $\delta = \gamma_s\alpha_0^{\sigma^s} - \alpha_0\gamma_s^{\sigma^{j_0}}$. If $\overline{\delta} \notin \mathbb{F}$, then $y_\gamma x y_\gamma^{-1} \equiv x(1+\alpha_0^{\sigma^{-j_0}}\delta^{\sigma^{-j_0}}\varpi^s) \pmod{1+P^{s+1}} \notin H''$ (recall that $s < m''$), and we are done.

Now suppose that $\overline{\delta} \in \mathbb{F}$. Since $x \in E$, we have $\alpha_0 \in \mathbb{F}^\sim$; thus

$$\delta = \alpha_0(\gamma_s - \gamma_s^{\sigma^{j_0}}); \qquad \operatorname{Tr}_{\mathbb{D}/\mathbb{F}}\delta = 0.$$

That is, $\delta \in T^\sim$. Thus, $\delta = 0$, so $\gamma_s = \gamma_s^{\sigma^{j_0}}$. But, since $\ell \nmid j_0$, σ^{j_0} generates $\operatorname{Gal}(\mathbb{D}/\mathbb{F})$; hence $\gamma_s \in \mathbb{F}$. As $\gamma_s \in T^\sim$, we have $\gamma_s = 0$. This contradicts the choice of γ_s, as desired. □

PROPOSITION 3.3. *Let* $x \in E$ *be such that* $m > \nu_0(x) = j > 0$, *and let* i *be the smallest integer with* $2i + j \geq m$. *Set*

$$(T^\sim)^{m''-1}(i) = \left\{ \gamma = (\gamma_1, \dots, \gamma_{m''-1}) \in (T^\sim)^{m''-1} \;\middle|\; \gamma_s = 0 \text{ for } s < i \right\}.$$

Then $\theta_\rho(x) = \sum_{\gamma \in (T^\sim)^{m''-1}(i)} \dot\theta_{\psi^\sim}(y_\gamma x y_\gamma^{-1})$.

REMARK. For $\gamma \in (T^\sim)^{m''-1}(i)$, since $y_\gamma \equiv 1 \pmod{P^{i-1}}$, we have $y_\gamma x y_\gamma^{-1} \equiv x \pmod{P^{j+i-1}}$. But $j+i \geq m'$, so that $y_\gamma x y_\gamma^{-1} \in H''$.

PROOF. Let γ be a nonzero element of $(T^\sim)^{m''-1}$, and let s be the smallest index such that $\gamma_s \neq 0$. If $j+s < m''$, then an argument like that in Proposition 3.2 gives $y_\gamma x y_\gamma^{-1} \notin H''$. So we may assume that $j+s \geq m''$.

Suppose that $m'' \le j+s < m-s$. (The latter inequality holds whenever $s < i$.) Let $k = m - j - s > s$, and consider

$$\sum_{\delta \in \mathbb{D}} \theta_{\psi^\sim}(z_\delta y_\gamma x y_\gamma^{-1} z_\delta^{-1}), \qquad z_\delta = 1 + \delta\varpi^k. \tag{3.4}$$

A bit of thought shows that the proposition holds if this sum is always 0 when $s < i$. So we examine (3.4). Dividing by an element of $F^\times$ if necessary, we may assume that

$$x = 1 + \alpha_j \varpi^j + \cdots, \qquad \ell \nmid j.$$

Then

$$x' = y_\gamma x y_\gamma^{-1} \equiv x + \varepsilon\varpi^{j+s} \pmod{P^{j+s+1}},$$
$$\varepsilon = \gamma_s \alpha_j^{\sigma^s} - \alpha_j \gamma_s^{\sigma^j} = \alpha_j(\gamma_s - \gamma_s^{\sigma^j}).$$

Hence

$$z_\delta x' z_\delta^{-1} \equiv x' + (\delta\varpi^k x - x\delta\varpi^k) + (\delta\varepsilon^{\sigma^k} - \varepsilon\delta^{\sigma^{s+j}})\varpi^m \pmod{P^{m+1}}.$$

That is,

$$z_\delta x' z_\delta^{-1} \equiv (x' + \delta\varpi^k x - x\delta\varpi^k)(1 + (\delta\varepsilon^{\sigma^k} - \varepsilon\delta^{\sigma^{s+j}})\varpi^m) \pmod{1 + P^{m+1}}.$$

We now examine $x' + \delta\varpi^k x - x\delta\varpi^k$. First of all,

$$1 + \delta\varpi^k x - x\delta\varpi^k \in K.$$

Why? Note first of all that lowest order term in $\delta\varpi^k x - x\delta\varpi^k = \delta\varpi^k(x-1) - (x-1)\delta\varpi^k$ is $\delta\varpi^k\alpha_j\varpi^j - \alpha_j\varpi^j\delta\varpi^k = \alpha_j(\delta - \delta^{\sigma^j})\varpi^{j+k}$, because $\alpha_i \in \mathbb{F}^\sim$, and $j + k \ge m'$.

Note secondly that if $u \in E$, then

$$\mathrm{Tr}_{D/F}(\delta\varpi^k x - x\delta\varpi^k)u = \mathrm{Tr}_{D/F}(\delta\varpi^k ux - ux\delta\varpi^k) = 0$$

because $\mathrm{Tr}_{D/F}(ab) = \mathrm{Tr}_{D/F}(ba)$ and $xu = ux$.

Thirdly, note that $x \equiv x' \pmod{P^{j+s}}$. Thus, setting $w = \delta\varpi^k x - x\delta\varpi^k$, we have

$$x' + w = x'(1+w) - (x'-1)w \equiv x'(1+w) - (x-1)w \pmod{P^{m+1}}.$$

But $x - 1 \in P \cap E$, and so it is easy to check that $1 + (x-1)w \in K$. Thus

$$x' + w \equiv x'(1+w)(1-(x-1)w) - (x-1)^2 w \pmod{P^{m+1}},$$

and it follows by induction that for some $w' \in K$,

$$x' + w \equiv x'w' \pmod{1 + P^{m+1}}. \tag{3.5}$$

It follows that $\theta_{\psi^\sim}(x' + w) = \theta_{\psi^\sim}(x')$. (Note that w depends on δ.) Thus we conclude that

$$\theta_{\psi^\sim}(z_\delta x' z_\delta^{-1}) = \theta_{\psi^\sim}(x')(\psi_0 \circ \mathrm{Tr}_{\mathbb{D}/\mathbb{F}}(\delta\varepsilon^{\sigma^k} - \varepsilon\delta^{\sigma^{s+j}})).$$

The sum in (3.4) is therefore a multiple of $\sum_{\delta\in\mathbb{D}} \psi_0 \circ \mathrm{Tr}_{\mathbb{D}/\mathbb{F}}(\delta\varepsilon^{\sigma^k} - \varepsilon\delta^{\sigma^{s+j}})$. Hence the sum is 0 unless for all δ,

$$1 = \psi_0 \circ \mathrm{Tr}_{\mathbb{D}/\mathbb{F}}(\delta\varepsilon^{\sigma^k} - \varepsilon\delta^{\sigma^{s+j}}) = \psi_0 \circ \mathrm{Tr}_{\mathbb{D}/\mathbb{F}}(\varepsilon^{\sigma^m} - \varepsilon).$$

As usual, this implies that $\varepsilon^{\sigma^m} = \varepsilon$, and hence that $\varepsilon \in \mathbb{F}$. That is,

$$\alpha_j(\gamma_s - \gamma_s^{\sigma^j}) \in \mathbb{F}.$$

Since $\alpha_j \in \mathbb{F}$, we also have $\alpha_j(\gamma_s - \gamma_s^{\sigma^j}) \in T$. Thus $\alpha_j(\gamma_s - \gamma_s^{\sigma^j}) = 0$, or $\gamma_s = \gamma_s^{\sigma^j}$. As σ^j generates $\mathrm{Gal}(\mathbb{D}/\mathbb{F})$, this means that $\gamma_s \in \mathbb{F}$. By construction, $\gamma_s \in T$. So $\gamma_s = 0$, a contradiction; the proposition is proved. □

PROPOSITION 3.6. *For $x \in E$ with $0 < \nu_0(x) = j < m$, and with i as in Proposition 3.3,*

(a) If $2i + j > m$, then $\theta_\rho(x) = q^{(\ell-1)(m''-i)}\theta_{\psi^\sim}(x)$;
(b) If $2i + j = m$, then

$$\theta_\rho(x) = q^{(\ell-1)(m''-i-1)}\theta_{\psi^\sim}(x) \sum_{\gamma \in T^\sim} \psi_0 \circ \mathrm{Tr}_{\mathbb{D}/\mathbb{F}}\, \alpha_j \gamma^{\sigma^{i+j}}(\gamma^{\sigma^j} - \gamma).$$

PROOF. We have

$$\begin{aligned} y_\gamma x y_\gamma^{-1} &= x + \sum_{d=1}^{\infty} (-1)^d \big[(x-1)(y_\gamma - 1) - (y_\gamma - 1)(x-1)\big](y-1)^{d-1} \\ &= x + \sum_{d=1}^{\infty} w_d \qquad \text{say.} \end{aligned}$$

Since $y_\gamma^{-1} \in P^i$ and $x - 1 \in P^j$, the w_ℓ with $\ell > 2$ are all in P^{m+1}.

(a) If $2i + j > m$, then $w_2 \in P^{m+1}$ as well. It is easy to check that $\mathrm{Tr}_{D/F}(w_1 z) = 0$ for all $z \in E$ (the argument is the same as one in Proposition 3.3), and $w_1 \in P^{m'}$ as well. Thus $\sum_{j=1}^{\infty} w_j \in K_0$, and an argument like that leading to (3.5) gives

$$y_\gamma x y_\gamma^{-1} = xk, \qquad k \in K.$$

Hence $\theta_{\psi^\sim}(y_\gamma x y_\gamma^{-1}) = \theta_{\psi^\sim}(x)$. Now (a) follows from the fact that

$$\mathrm{card}\, T^{m''-1}(i) = q^{(m''-i)(\ell-1)}.$$

(b) If $2i + j = m$, then a calculation gives

$$w_2 - \alpha_j \gamma_i^{\sigma^{i+j}}(\gamma_i^{\sigma^j} - \gamma_i)\varpi^m \in P^{m+1}.$$

(Only the lowest order terms in the expression for w_2 are not in P^{m+1}, so that the calculation is not too hard.) But this means that $\pmod{1 + P^{m+1}}$,

$$y_\gamma x y_\gamma^{-1} = (x + w_1)(1 + \alpha_j \gamma_i^{\sigma^{i+j}}(\gamma_i^{\sigma^j} - \gamma_i)\varpi^m).$$

As with (a), $x + w_1 = xk$, $k \in K$. It is now easy to compute $\theta_{\psi^\sim}(y_\gamma x y_\gamma^{-1})$; summing yields (b) (with γ_i for γ). □

We will have an estimate for $|\theta_\rho(x)|$ once we evaluate the sum in (b).

PROPOSITION 3.7. *If $\nu_0(x) < m$ and x is (conjugate to) an element of $E^\times$, then*

$$\theta_\rho(x) = q^{\frac{1}{2}(\ell-1)\nu_0(x)} G(x, \psi)\psi(x),$$

where $G(x, \psi)$ is a fourth root of unity. Indeed,

$$G(x,\psi) = \begin{cases} \lambda(E/F) & \text{if } \nu_0(x) = 0; \\ 1 & \text{if } j > 0, \text{ and } m \text{ and } j \text{ have opposite parities;} \\ q^{-(\ell-1)/2} \sum_{\gamma \in T} \psi_0 \circ \mathrm{Tr}_{\mathbb{D}/\mathbb{F}}(\alpha \gamma^{\sigma^{i+j}}(\gamma^{\sigma^j} - \gamma)) & \\ & \text{if } j > 0, \text{ and } m \text{ and } j \text{ have the same parity.} \end{cases} \tag{3.8}$$

PROOF. Without loss of generality, we may assume x normal. We first consider the case when $\nu_0(x) = 0$. Here, by Proposition 3.2 and Fact 3 (near the end of §1) we have $\theta_\rho(x) = \theta_{\psi^\sim}(x) = \lambda(E/F)\psi(x)$.

Next, we consider $\nu_0(x) > 0$.

(i) If m is odd and $\nu_0(x) = j$ is even, then Proposition 3.6(a) applies. We have $\theta_{\psi^\sim}(x) = \psi^\#(x)$, $m'' = \frac{m+1}{2}$, and $i = \frac{m+1-j}{2}$. Hence the exponent of q in Proposition 3.6(a) is

$$(\ell-1)(m''-i) = (\ell-1)(\tfrac{m+1}{2} - \tfrac{(m+1-j)}{2}) = \tfrac{1}{2}(\ell-1)j,$$

as desired. Here, $G(x,\psi) = 1$.

(ii) Suppose m is even and $\nu_0(x) = j$ is odd. The hypothesis that m is even implies by Fact 3 (near the end of §1) that $m'' = \frac{m}{2}$ and $\theta_{\psi^\sim}(x) = q^{(\ell-1)/2}\psi(x)$. The j odd hypothesis means $i = \frac{m+1-j}{2}$ and therefore we are again in the situation of Proposition 3.6(a). Hence

$$\theta_\rho(x) = \theta_\psi(x)q^a \qquad (G(x,\psi) = 1),$$

where

$$a = \tfrac{\ell-1}{2} + (\ell-1)(m''-i) = \tfrac{\ell-1}{2} + (\ell-1)(\tfrac{m}{2} - \tfrac{m+1-j}{2}) = \tfrac{\ell-1}{2}j$$

again.

(iii) If m and j are both odd, we use Proposition 3.6(b) and Proposition 3.9. Since $\theta_{\psi^\sim} = \psi$, we see that

$$\theta_\rho(x) = \psi(x)G(x,\psi)q^a,$$

where, since $m'' = \frac{m+1}{2}$ and $i = \frac{m-j}{2}$,

$$a = \tfrac{\ell-1}{2} + \tfrac{\ell-1}{2}(m+1-(m-j)-2) = \tfrac{\ell-1}{2}j$$

again.

(iv) If m and j are both even, we need Proposition 3.6(b), Proposition 3.9, and the fact that $\theta_{\psi^\sim}(x) = q^{(\ell-1)/2}\psi(x)$. The algebra is routine.

That $G(x,\psi)$ is a fourth root of unity is clear in the first two cases of (3.8). The proof in the third case of (3.8) is the next proposition. □

PROPOSITION 3.9. *If q is odd, then*

$$\sum_{\gamma\in T} \psi_0 \circ \mathrm{Tr}_{\mathbb{D}/\mathbb{F}}(\alpha\gamma^{\sigma^{i+j}}(\gamma^{\sigma^j} - \gamma)) = G(x,\psi)q^{(\ell-1)/2},$$

where $G(x,\psi)$ is a fourth root of unity.

PROOF. Set

$$Q(\gamma,\delta) = \mathrm{Tr}_{\mathbb{D}/\mathbb{F}}\big[\gamma^{\sigma^{i+j}}(\delta^{\sigma^j} - \delta) + \delta^{\sigma^{i+j}}(\gamma^{\sigma^j} - \gamma)\big], \qquad \gamma,\delta \in T.$$

Then the sum we need to evaluate is

$$\sum_{\gamma\in T} \psi_0(\tfrac{1}{2}\alpha_j Q(\gamma,\gamma)). \tag{3.10}$$

By results going back to Gauss, (3.5) is $(\mathrm{card}\, T)^{\frac{1}{2}}G(x,\psi)$, where $G(x,\psi)$ is a fourth root of unity, provided that Q is nondegenerate. Rewrite Q as

$$Q(\gamma,\delta) = \mathrm{Tr}_{\mathbb{D}/\mathbb{F}}\,\delta(\gamma^{\sigma^i} - \gamma^{\sigma^{i+j}} + \gamma^{\sigma^{-i}} - \gamma^{\sigma^{-i-j}}).$$

Suppose that for some $\gamma \in T$, $Q(\gamma, \delta) = 0$ for all $\delta \in T$. Obviously $Q(\gamma, \delta) = 0$ if $\delta \in \mathbb{F}$; since $\mathbb{D} = \mathbb{F} \oplus T$, we get

$$\gamma^{\sigma^i} - \gamma^{\sigma^{i+j}} + \gamma^{\sigma^{-i}} - \gamma^{\sigma^{-i-j}} = 0.$$

Using the fact that $m = 2i + j$, we can rewrite this as

$$(\gamma - \gamma^{\sigma^j})^{\sigma^m} = (\gamma - \gamma^{\sigma^j}).$$

Since m and ℓ are relatively prime, it follows that $\gamma - \gamma^{\sigma^j} \in \mathbb{F}$.

But obviously $\gamma - \gamma^{\sigma^j} \in T$; thus $\gamma - \gamma^{\sigma^j} = 0$. Moreover, since σ^j generates $\mathrm{Gal}(\mathbb{D}/\mathbb{F})$, we have $\gamma \in \mathbb{F}$. But $\gamma \in T$; hence $\gamma = 0$. This proves nondegeneracy. □

4. Computing Θ_π explicitly

The following theorem summarizes previous results.

THEOREM 4.1. *For $x \in H$ and q odd,*

(a) If $\nu_0(x) \geq m$, then $\theta_\rho(x) = q^{\frac{1}{2}(m-1)(\ell-1)}\psi^\#(x)$.

(b) If $\nu_0(x) < m$ and x is not (conjugate to) an element of E, then $\theta_\rho(x) = 0$.

It is fairly easy to compute Θ_π from θ_ρ.

THEOREM 4.2. *Let $x \in D^\times$.*

(a) If $\nu_0(x) > m$, then $\Theta_\pi(x) = \frac{q^\ell - 1}{q-1} q^{\frac{1}{2}(m-1)(\ell-1)}\psi^\#(x)$.

(b) If $\nu_0(x) < m$ and x is not conjugate to an element of E, then $\Theta_\pi(x) = 0$.

(c) If $\nu_0(x) < m$ and $x \in E$, then

$$\Theta_\pi(x) = q^{\frac{1}{2}(m-1)\nu_0(x)} \sum_{y \in (\mathrm{Gal}(E/F))\cdot x} G(y, \psi)\psi(y),$$

where $\mathrm{Gal}(E/F)$ is the group of all automorphisms of E fixing F. (In our case, it is $\{1\}$ unless E is normal, since there are no intermediate fields.)

(d) If $\nu_0(x) = m$, then $x = x_0 f$, $f \in F$ and $x_0 \equiv 1 + \alpha\varpi^m \pmod{P^{m+1}}$. Then

$$\Theta_\pi(x) = q^{\frac{1}{2}(m-1)(\ell-1)}\psi(f) \sum_{\substack{\beta \in \mathbb{D}^\times \\ \mathrm{N}_{\mathbb{D}/\mathbb{F}}(\beta/\alpha) = 1}} \psi_0 \circ \mathrm{Tr}_{\mathbb{D}/\mathbb{F}}(\beta).$$

(Note that $\alpha \neq 0$, since $\nu_0(x) = m$.)

PROOF. The main point to notice is that H is normal in $D^\times$, with $D^\times/H \cong \mathbb{D}^\times/\mathbb{F}^\times$. The rest is just a calculation with the Frobenius formula

$$\Theta_\pi(x) = \sum_{w \in D^\times/H} \dot{\theta}_\rho(wxw^{-1}). \tag{4.3}$$

(a) If $\nu_0(x) > m$, then $wxw^{-1} \in \ker\rho$ for all $w \in D^\times/H$, and $\mathrm{card}\, D^\times/H = \frac{q^\ell - 1}{q-1}$. (Note that $\psi^\#(x) = 1$; the only point of including it is to make the formula valid after ψ is twisted by a character of F.)

(b) This is immediate from (b) of Theorem 4.1.

(c) The only terms in (4.3) that contribute to the sum nontrivially are those for which $wxw^{-1} \in H$. In fact, by Theorem 4.1 we can assume $wxw^{-1} \in E$. Since x generates E, if if $wxw^{-1} \in E$, then conjugation by w gives an automorphism of E. Furthermore, if conjugation by two elements w and w' yield the same automorphism of E, the fact that E/F is tamely ramified implies w and w' lie

in the same H coset in $D^\times$. Finally, the Skolem-Noether Theorem says that every automorphism of E fixing F comes from an inner automorphism of D. Now (c) is clear.

(d) Obviously we may restrict to the case $f = 1$. Choose the w to be in $\mathbb{D}^{\sim\times}$ (mod $\mathbb{F}^{\sim\times}$); then (mod $1 + P^{m+1}$), if $w = \gamma$,

$$wx_0w^{-1} = 1 + \gamma\alpha(\gamma^{\sigma^m})^{-1}\varpi^m.$$

As γ^m generates $\mathrm{Gal}(\mathbb{D}/\mathbb{F})$, the elements $\gamma_\alpha(\gamma^{\sigma^m})^{-1}$ are precisely the elements β of $\mathbb{D}$ with $\mathrm{N}_{\mathbb{D}/\mathbb{F}}(\beta/\alpha)$, each occurring once. That, with (4.3), gives the proof. □

NOTE. Of course, if x is conjugate to $x' \in E$, then $\Theta_\pi(x) = \Theta_\pi(x')$. Thus we have computed Θ_π everywhere in D.

References

[1] L. Corwin, *Representations of division algebras over local fields*, Adv. in Math. **13** (1974), 259–267.

[2] L. Corwin and R. Howe, *Computing characters of tamely ramified p-adic division algebras*, Pac. J. Math **73** (1977), no. 2, 461–477.

[3] L. Corwin, A. Moy, and P. J. Sally, Jr., *Discrete series characters for division algebras and $GL_n(F)$, F p-adic*, in preparation.

[4] ———, *Degrees and formal degrees for division algebras and GL_n over a p-adic field*, Pac. J. Math. **141** (1990), no. 1, 21–45.

[5] P. Gérardin, *Sur les représentations du groupe linéaire general sur un corps p-adique*, Séminaire Delange-Pisot-Poitou, Théorie des nombres, Fasc. 1, No. 12, Secretariat Mathematique, 1972/73.

[6] T. Hales, *Hyperelliptic curves and harmonic analysis*, Representation Theory and Analysis on Homogeneous Spaces: A Conference in Memory of Larry Corwin (S. Gindikin, R. Goodman, F. Greenleaf, and P. J. Sally, Jr., eds.), Contemporary Mathematics, vol. 177, American Mathematical Society, 1994, pp. 137–170.

[7] R. Howe, *Representation theory for division algebras over local fields (tamely ramified case)*, Bull. Amer. Math. Soc. **77** (1971), 1063–1066.

[8] ———, *Tamely ramified supercuspidal representations of GL_n*, Pac. J. Math. **73** (1977), no. 2, 437–460.

[9] P. Kutzko, *Character formulas for supercuspidal representations of GL_l, l a prime*, Amer. J. Math. **109** (1987), no. 2, 201–221.

[10] A. Moy, *Local constants and the tame Langlands correspondence*, Amer. J. Math. **108** (1986), 863–930.

[11] P. J. Sally, Jr., *The work of Lawrence Corwin on the representation theory of reductive p-adic groups*, Representation Theory and Analysis on Homogeneous Spaces: A Conference in Memory of Larry Corwin (S. Gindikin, R. Goodman, F. Greenleaf, and P. J. Sally, Jr., eds.), Contemporary Mathematics, vol. 177, American Mathematical Society, 1994, pp. 59–64.

[12] P. J. Sally, Jr. and J. Shalika, *The Plancherel formula for $SL(2)$ over a local field*, Proc. Nat. Acad. Sci. U. S. A. **63** (1969), 661–667.

[13] H. Shimizu, *Some examples of new forms*, J. Fac. Sci. U. Tokyo **24** (1977), 97–113.

[14] Allan J. Silberger, *PGL_2 over the p-adics: its representations, spherical functions, and Fourier analysis*, Lecture Notes in Mathematics, vol. 166, Springer-Verlag, 1970.

[15] A. Weil, *Basic number theory*, Springer, 1974.

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Geometric Realizations for Highest Weight Representations

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Dedicated to Ray A. Kunze

ABSTRACT. Let ω be an irreducible unitary representation of a Hermitian group G having a highest weight vector. In this paper we construct an antiholomorphic mapping from the bounded symmetric domain G/K into the space of linear operators from the K linear span H of the highest weight vector into the Hilbert space of the unitary representation ω. This operator is used to obtain the intertwining operator and the unitary structure for the geometric realization of ω.

1. Introduction

The theory of highest weight unitary representations is well-developed algebraically. For Hermitian groups they are described in terms of Verma modules and conditions for unitarity are determined by specific algebraic conditions, [5]. This article will address how these representations can be naturally realized analytically on a Hilbert space of holomorphic functions on complex bounded domains associated with the group. In particular we provide an abstract method for obtaining an intertwining operator between any highest weight representation and its geometric realization. This is done by determining a natural dense subspace of analytic vectors on which the action of the group is given by a composition of an operator valued multiplier and the natural action on the bounded symmetric domain. In the literature there are several papers, [1, 2, 9, 10], where concrete realizations are considered that fit into this framework.

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For a unitary highest weight representation ω of an Hermitian symmetric group G we construct an antiholomorphic mapping $T \mapsto q_T$ on the bounded domain $\mathcal{D} = G/K$ into the set of operators from the irreducible K space H consisting of the K span of the highest weight vector for ω into the Hilbert space $\mathbb{H}$ of the representation ω. The adjoints q_T^* of these operators are then used to obtain a unitary isomorphism of $\mathbb{H}$ into a Hilbert space of holomorphic H valued functions on $\mathcal{D}$. Specifically, this mapping is defined by $v \mapsto \Xi(v)$ where $\Xi(v)(T) = q_T^* v$. Furthermore, $Q(T,S) = q_T^* q_S$ defines a positive definite operator valued function on $\mathcal{D}$, and this positive definite function when used in the sense of Kunze, [8], gives a Hilbert space of holomorphic functions on $\mathcal{D}$ on which $Q(T,S)$ is the reproducing kernel. This space is precisely the range of Ξ. In particular Ξ gives a geometric realization of the representation ω on a space of holomorphic vector valued functions on $\mathcal{D}$. It is shown this realization can be completely described using the operators q_T and a multiplier $J(g,T)$ which is defined in terms of the irreducible representation of K on H.

The operator q_T is constructed using a power series involving the action of the conjugate of T relative to the infinitesimal representation of ω. For nonzero v in H, it is shown in Theorem 5.1 that the series for $q_T v$ converges iff T is in the domain $\mathcal{D}$. Thus existence of the operator q_T corresponds exactly to T being located in the domain $\mathcal{D} = G/K$. The operators q_T are related to the representation ω in Theorem 7.1 by the formula

$$\omega(g) q_T v = q_{g \cdot T} J(g,T)^{*-1} v.$$

This is used in showing Ξ intertwines the representation ω with the geometric representation $(\mathcal{T}, \mathcal{H}(D,H))$ where $\mathcal{H}(D,H)$ is the Hilbert space determined by $Q(S,T)$ and $\mathcal{T}$ is the natural geometric representation on this space.

2. Notation and Preliminaries

Let G denote a connected noncompact simple Lie group with finite center. Let K be a maximal compact subgroup. We assume K has a one-dimensional center. This implies that G/K is a Hermitian symmetric space. Let $T \subset K$ be a compact Cartan subgroup and let $\mathfrak{g}_o$, $\mathfrak{k}_o$, and $\mathfrak{t}_o$ be the Lie algebras of G, K, and T respectively. Take $\mathfrak{g}_o = \mathfrak{k}_o + \mathfrak{p}_o$ to be the Cartan decomposition. We adopt the convention that the removal of the subscript o will denote complexification. Our assumptions imply that there exists a decomposition $\mathfrak{g} = \mathfrak{p}_+ \oplus \mathfrak{k} \oplus \mathfrak{p}_-$ where $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are abelian subalgebras of $\mathfrak{p}$ satisfying

$$\begin{aligned} &[\mathfrak{k}, \mathfrak{p}_\pm] \subset \mathfrak{p}_\pm \text{ and} \\ &[\mathfrak{p}_+, \mathfrak{p}_-] \subset \mathfrak{k}. \end{aligned} \tag{2.1}$$

Furthermore, if a bar $\bar{\ }$ denotes conjugation on $\mathfrak{g}$ with respect to $\mathfrak{g}_o$, then we have

$$\bar{\mathfrak{p}}_\pm \subset \mathfrak{p}_\mp. \tag{2.2}$$

The subalgebra $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. We let Φ be the roots corresponding to the pair $(\mathfrak{g}, \mathfrak{t})$. Let Φ_c and Φ_n denote the set of compact and noncompact roots, respectively. We denote the root space corresponding to $\alpha \in \Phi$ by $\mathfrak{g}_\alpha$. A positive system of roots Φ^+ may be chosen so that if $\Phi_n^+ = \Phi^+ \cap \Phi_n$ and $\Phi_n^- = (-\Phi^+) \cap \Phi_n$ then

$$\begin{aligned} \mathfrak{p}_+ &= \oplus_{\alpha \in \Phi_n^+} \mathfrak{g}_\alpha \text{ and} \\ \mathfrak{p}_- &= \oplus_{\alpha \in \Phi_n^-} \mathfrak{g}_\alpha. \end{aligned} \tag{2.3}$$

If θ denotes the Cartan involution on $\mathfrak{g}$ and B denotes the Killing form, then one can choose root vectors $E_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Phi$, so that $B(E_\alpha, E_{-\alpha}) = 2/(\alpha, \alpha)$, and $\theta \bar{E}_\alpha = -E_{-\alpha}$, where $(\cdot, \cdot)$ is the standard inner product on the real space of linear functionals on $\mathfrak{t}$ taking purely imaginary values on $\mathfrak{t}_\circ$. We put $H_\alpha = [E_\alpha, E_{-\alpha}]$. Then we have

$$\begin{aligned} [H_\alpha, E_\alpha] &= 2E_\alpha \text{ and} \\ [H_\alpha, E_{-\alpha}] &= -2E_{-\alpha}. \end{aligned}$$

Let r be the split rank of G. We inductively define a maximal set $\{\gamma_1, \ldots, \gamma_r\}$ of strongly orthogonal positive noncompact roots by setting γ_1 to be the largest root of Φ^+ and γ_{j+1} the largest root in Φ_n^+ which is strongly orthogonal to $\gamma_1, \cdots, \gamma_j$. Then $E_{\gamma_j} + E_{-\gamma_j}$ lies in $\mathfrak{p}_\circ$ for each $1 \leq j \leq r$ and

$$\mathfrak{a}_\circ = \bigoplus_{1 \leq j \leq r} \mathbb{R}(E_{\gamma_j} + E_{-\gamma_j}) \tag{2.4}$$

is a maximal abelian subspace of $\mathfrak{p}_\circ$. Let A be the connected subgroup of G whose Lie algebra is $\mathfrak{a}_\circ$. Then we have the well known decomposition $G = KAK$.

We now turn our attention to the Harish-Chandra realization of G/K as a bounded symmetric domain in $\mathfrak{p}_+$. For this it is convenient to assume that G has a faithful representation; thus $G \subset G_{\mathbb{C}}$. Let $P_\pm = \exp(\mathfrak{p}_\pm)$. Then the mapping $\mathfrak{p}_+ \times K_{\mathbb{C}} \times \mathfrak{p}_- \to G_{\mathbb{C}}$ defined by $(a, k, b) \mapsto \exp(a) k \exp(b)$ is a holomorphic diffeomorphism onto a dense open set $\Omega = P_+ K_{\mathbb{C}} P_-$ of $G_{\mathbb{C}}$. We uniquely write each $x \in \Omega$ as a product

$$x = \pi_+(x) \pi_\circ(x) \pi_-(x), \tag{2.5}$$

where $\pi_\pm(x) \in P_\pm$ and $\pi_\circ(x) \in K_{\mathbb{C}}$. It is well-known that $G \subset \Omega$ and that the map $\zeta : \Omega \to \mathfrak{p}_+$ defined by $\zeta(x) = \log(\pi_+(x))$ induces a holomorphic diffeomorphism of G/K onto $\zeta(G)$. The set $\mathcal{D} = \zeta(G)$ is a bounded symmetric domain in $\mathfrak{p}_+$ which we identify with G/K.

For each $(g, T) \in G_{\mathbb{C}} \times \mathfrak{p}_+$ for which $g \exp(T) \in \Omega$ we set

$$\begin{aligned} g \cdot T &= \log \pi_+(g \exp(T)) \\ j(g, T) &= \pi_\circ(g \exp(T)). \end{aligned} \tag{2.6}$$

The map $j : G_{\mathbb{C}} \times \mathfrak{p}_+ \to K_{\mathbb{C}}$ is a C^∞ map and is holomorphic in the $\mathfrak{p}_+$ variable. Furthermore, it satisfies the following properties

$$\begin{aligned} j(k,T) &= k, \\ j(p,T) &= 1, \\ j(g_1 g_2, T) &= j(g_1, g_2 \cdot T) j(g_2, T), \end{aligned} \tag{2.7}$$

where $k \in K_{\mathbb{C}}$, $T \in \mathfrak{p}_+$, $p \in P_+$, $g_1, g_2 \in G$, and for which $g_1 g_2 \exp(T)$ and $g_2 \exp(T)$ are in Ω.

3. Highest Weight Representations

Let ω be a nontrivial irreducible unitary highest weight representation of G on a Hilbert space $\mathbb{H}$ with inner product denoted by $\langle \cdot | \cdot \rangle$. We will let $d\omega$ denote both the derived representation of ω on $\mathfrak{g}_o$ and its complex linear extension to $\mathfrak{g}$. However, when there is no confusion we will usually write $X\phi$ or $X \cdot \phi$ instead of $d\omega(X)\phi$, where ϕ is a C^∞ vector and X is in the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$. Let v_o denote the highest weight vector and let $H = \mathbb{H}_o$ denote the K span of v_o. Then H is an irreducible K-space. Let τ denote the restriction of $\omega|_K$ to H and set λ to be its highest weight. By (2.1) we also have

$$T \cdot v = 0 \tag{3.1}$$

for all $T \in \mathfrak{p}_+$ and $v \in H$. For each positive integer n we define $\mathbb{H}_n$ to be the subspace of $\mathbb{H}$ spanned by all terms of the form $\bar{E}^{\mathbf{r}} v$, where $v \in H$, r is a multi-index with $\sum_{\alpha \in \Phi_n^+} r_\alpha = n$, and $\bar{E}^{\mathbf{r}} = \prod_{\alpha \in \Phi_n^+} \bar{E}_\alpha^{r_\alpha}$. By formulas (2.1) and (2.2) it is easy to verify that

$$\begin{aligned} x &: \mathbb{H}_n \to \mathbb{H}_{n+1} \text{ for } x \in \mathfrak{p}_-, \\ x &: \mathbb{H}_n \to \mathbb{H}_n \text{ for } x \in \mathfrak{k}, \\ x &: \mathbb{H}_n \to \mathbb{H}_{n-1} \text{ for } x \in \mathfrak{p}_+, \end{aligned} \tag{3.2}$$

for each nonnegative integer n. The space $\mathbb{H}_{-1}$ is understood to be the trivial space. From this it follows that $\bigoplus_{n \geq 0} \mathbb{H}_n$ is a $\mathfrak{g}$-invariant subspace of $\mathbb{H}$ which is dense by irreducibility.

REMARK 3.1. *Since, by unitarity, the adjoint of $d\omega(X)$ is $-d\omega(\bar{X})$ formulas (3.2) imply that the sum $\bigoplus_{n \geq 0} \mathbb{H}_n$ is an orthogonal direct sum.*

We now define an operator valued multiplier $J = J_\tau$ in terms of τ and the K valued multiplier j defined by equation (2.6). Let $(g,T) \in G_{\mathbb{C}} \times \mathfrak{p}_+$, for which $g \exp(T) \in \Omega$. We define $J(g,T) = \tau(j(g,T))$. Since τ is a homomorphism, it follows from equations (2.7) that

$$\begin{aligned} J(k,T) &= \tau(k), \\ J(p,T) &= I, \text{ and} \\ J(g_1 g_2, T) &= J(g_1, g_2 \cdot T) J(g_2, T). \end{aligned} \tag{3.3}$$

4. Three-Dimensional Systems

We develop some results for representations of three-dimensional systems having highest weight vectors. We assume G is a real Lie group and ρ is a unitary representation of G. If $H, E, \bar{E}$ are in $\mathfrak{g}$, the complexification of the Lie algebra of G, and if $[E, \bar{E}] = H$, $[H, E] = 2E$, $[H, \bar{E}] = -2\bar{E}$, then the triple $\{H, E, \bar{E}\}$ is a three-dimensional system. We say v is a highest weight vector for the three-dimensional system $\{H, E, \bar{E}\}$ if v is an analytic vector for the representation ρ, $d\rho(H)v = \mu v$ and $d\rho(E)v = 0$. As before we use the notation $Y \cdot w$ or Yw to denote $d\rho(Y)w$ for C^∞ vectors w and vectors $Y \in \mathcal{U}(\mathfrak{g})$.

LEMMA 4.1. *Let $\{H, E, \bar{E}\}$ be a three-dimensional system. Then*

(a) $[H, \bar{E}^n] = -2n\bar{E}^n$.
(b) $[H, E^n] = 2nE^n$.
(c) $E\bar{E}^n = \bar{E}^n E + n\bar{E}^{n-1}H - n(n-1)\bar{E}^{n-1}$.

PROOF. Note $[H, \bar{E}] = -2\bar{E}$ gives $H\bar{E} = \bar{E}H - 2\bar{E}$. Thus

$$\begin{aligned} H\bar{E}^2 &= \bar{E}H\bar{E} - 2\bar{E}^2 \\ &= \bar{E}^2 H - 4\bar{E}^2. \end{aligned}$$

Hence suppose $H\bar{E}^n = \bar{E}^n H - 2n\bar{E}^n$. Then

$$\begin{aligned} H\bar{E}^{n+1} &= H\bar{E}^n\bar{E} \\ &= (\bar{E}^n H - 2n\bar{E}^n)\bar{E} \\ &= \bar{E}^n\bar{E}H - 2\bar{E}^{n+1} - 2n\bar{E}^{n+1} \\ &= \bar{E}^{n+1}H - 2(n+1)\bar{E}^{n+1}. \end{aligned}$$

This gives (1). A similar argument gives (2). For (3) first note $E\,\bar{E} = \bar{E}E + H$. Hence if $E\bar{E}^n = \bar{E}^n E + n\bar{E}^{n-1}H - n(n-1)\bar{E}^{n-1}$ we see

$$\begin{aligned} E\bar{E}^{n+1} &= (\bar{E}^n E + n\bar{E}^{n-1}H - n(n-1)\bar{E}^{n-1})\bar{E} \\ &= \bar{E}^{n+1}E + \bar{E}^n H + n\bar{E}^n H - 2n\bar{E}^n - n(n-1)\bar{E}^n \\ &= \bar{E}^{n+1}E + (n+1)\bar{E}^n H - (n+1)n\bar{E}^n. \end{aligned}$$

□

LEMMA 4.2. *Suppose v is a highest weight vector with weight μ for the three-dimensional system $\{H, E, \bar{E}\}$. Then*

(a) $E^m\bar{E}^n v = \frac{n!}{(n-m)!}\frac{\Gamma(\mu-n+m+1)}{\Gamma(\mu-n+1)}\bar{E}^{n-m}v$ *if* $m < n$.
(b) $E^n\bar{E}^n v = n!\frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)}v$.
(c) $E^m\bar{E}^n v = 0$ *if* $m > n$.

PROOF. By Lemma 4.1 $E\bar{E}^n v = n\bar{E}^{n-1}Hv - n(n-1)\bar{E}^{n-1}v = n(\mu - n + 1)\bar{E}^{n-1}v$. Suppose

$$E^k\bar{E}^n v = \frac{n!}{(n-k)!}\frac{\Gamma(\mu-n+k+1)}{\Gamma(\mu-n+1)}\bar{E}^{n-k}v.$$

Then again by Lemma 4.1

$$\begin{aligned} E^{k+1}\bar{E}^n v &= \frac{n!}{(n-k)!}\frac{\Gamma(\mu-n+k+1)}{\Gamma(\mu-n+1)}E\bar{E}^{n-k}v \\ &= \frac{n!}{(n-k)!}\frac{\Gamma(\mu-n+k+1)}{\Gamma(\mu-n+1)}((n-k)\bar{E}^{n-k-1}Hv \\ &\quad -(n-k)(n-k-1)\bar{E}^{n-k-1}v) \\ &= \frac{n!}{(n-k)!}\frac{\Gamma(\mu-n+k+1)}{\Gamma(\mu-n+1)}(n-k)(\mu-n+k+1)\bar{E}^{n-k-1}v \\ &= \frac{n!}{(n-k-1)!}\frac{\Gamma(\mu-n+k+2)}{\Gamma(\mu-n+1)}\bar{E}^{n-k-1}v. \end{aligned}$$

This gives (a) and (b). For (c) note

$$E^m\bar{E}^n v = E^{m-n}E^n\bar{E}^n v = E^{m-n}n!\frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)}v = 0.$$

□

5. The Operator q_T

In this section we define a linear operator $q_T : H \to \mathbb{H}$, for each $T \in \mathcal{D}$. Initially, it is defined in terms of a power series involving the exponential function and the derived representation $d\omega$. In the next section q_T will be related to the operations of the group representation.

Let $T \in \mathfrak{p}_+$. Define $q_T : H \to \mathbb{H}$ by the formula

$$q_T v = \sum_{n=0}^{\infty} \frac{(\bar{T})^n v}{n!}, \tag{5.1}$$

where $v \in H$. The next theorem shows that the convergence of this series in $\mathbb{H}$ completely characterizes $\mathcal{D}$.

THEOREM 5.1. *Let v be a nonzero vector in H and $T \in \mathfrak{p}_+$. Then the series that defines $q_T v$ converges in the Hilbert space $\mathbb{H}$ if and only if $T \in \mathcal{D}$.*

PROOF. Let $T \in \mathfrak{p}_+$ and $v \in H$. To determine the convergence of $q_T v$ it is enough by Remark 3.1 to determine when the sum $\sum (n!)^{-2}||\bar{T}^n v||^2$ converges. Let $a_+ = \{\sum_{j=1}^{r} t_j E_{\gamma_j} : 0 \le t_j < \infty\}$. Then each element T of $\mathfrak{p}_+$ can be written in the form $T = \mathrm{Ad}(k)c$ for some $c \in a_+$ and $k \in K$. Furthermore each T of $\mathcal{D}$ can be written in this same form with the restriction that $0 \le t_j < 1$, for each j

(Corollary 7.17 of [7]). By unitarity we have for such a T

$$\begin{aligned}\sum_{n\geq 0}\frac{1}{(n!)^2}||\bar{T}^n v||^2 &= \sum_{n\geq 0}\frac{1}{(n!)^2}||(\mathrm{Ad}(k)\bar{c})^n v||^2\\ &= \sum_{n\geq 0}\frac{1}{(n!)^2}||\omega(k)\bar{c}^n\tau(k)v||^2\\ &= \sum_{n\geq 0}\frac{1}{(n!)^2}||\bar{c}^n w||^2,\end{aligned}$$

where $w = \tau(k)v$. It follows that we only need to determine for which $c \in a_+$ the series for $q_c w$ converges. Furthermore, since each $w \in H$ can be written as a finite sum of weight vectors in H we may assume w is a unit weight vector. Let μ be the weight of w. We now concentrate on evaluating the expression $||\bar{c}^n w||^2 = (-1)^n\langle c^n\bar{c}^n w|w\rangle$. Write $c = \sum t_i E_i$, where $E_i = E_{\gamma_i}$, $1 \leq i \leq r$. Also let $H_i = H_{\gamma_i}$. Then

$$c^n = (\sum t_i E_i)^n = \sum_{|\alpha|=n}\frac{n!}{\alpha!}t^\alpha E^\alpha$$

where $t^\alpha = t_1^{\alpha_1}\cdots t_r^{\alpha_r}$ and $E^\alpha = E_1^{\alpha_1}\cdots E_r^{\alpha_r}$. Thus we get

$$\langle c^n\bar{c}^n w|w\rangle = \sum_{|\alpha|=n,|\beta|=n}\frac{n!}{\alpha!}\frac{n!}{\beta!}\quad t^\alpha\bar{t}^\beta\ \langle E^\alpha\bar{E}^\beta w|w\rangle.$$

Since $E_1,\cdots,E_r$ commute and $[E_i,\bar{E}_j] = 0$ if $i \neq j$ it follows that we can write $E^\alpha\bar{E}^\beta$ as a product of commuting operators $(E_1^{\alpha_1}\bar{E}_1^{\beta_1})\cdots(E_r^{\alpha_r}\bar{E}_r^{\beta_r})$. Now suppose $\alpha \neq \beta$. Then for some index i we have $\alpha_i > \beta_i$ and $\langle E_1^{\alpha_1}\bar{E}_1^{\beta_1}\cdots E_r^{\alpha_r}\bar{E}_r^{\beta_r}w|w\rangle = 0$ by Lemma 4.2 (c). It now follows that

$$\langle c^n\bar{c}^n w|w\rangle = \sum_\alpha(\frac{n!}{\alpha!})^2\, t^\alpha\bar{t}^\alpha\ \langle E^\alpha\bar{E}^\alpha w|w\rangle.$$

Furthermore, by Lemma 4.2 (b) we obtain

$$\begin{aligned}\langle c^n\bar{c}^n w|w\rangle &= \sum_\alpha(\frac{n!}{\alpha!})^2\, t^\alpha\bar{t}^\alpha\alpha_1!\cdots\alpha_r!\ \mu^{(\alpha_1)}(H_1)\cdots\mu^{(\alpha_r)}(H_r)\,\langle w|w\rangle\\ &= \sum_\alpha\frac{(n!)^2}{\alpha!}\,\mu^{(\alpha_1)}(H_1)\cdots\mu^{(\alpha_r)}(H_r)\,t^\alpha\bar{t}^\alpha\end{aligned}$$

where $a^{(r)} = a(a-1)\cdots(a-r+1) = \frac{\Gamma(a+1)}{\Gamma(a-r+1)}$. Putting this together we finally

obtain,

$$
\begin{aligned}
\langle q_c w | q_c w \rangle &= \sum_{n=0}^{\infty} \frac{1}{(n!)^2} ||\bar{c}^n w||^2 \\
&= \sum_n (-1)^n \frac{1}{(n!)^2} \langle c^n \bar{c}^n w | w \rangle \\
&= \sum_n (-1)^n \sum_{|\alpha|=n} \frac{\mu^{(\alpha_1)}(H_1) \cdots \mu^{(\alpha_r)}(H_r)}{\alpha!} t^\alpha \bar{t}^\alpha \\
&= \sum_\alpha (-1)^\alpha \frac{\mu^{(\alpha_1)}(H_1)) \cdots \mu^{(\alpha_r)}(H_r)}{\alpha!} t^\alpha \bar{t}^\alpha \qquad (5.2) \\
&= \prod_{i=1}^r \sum_{\alpha_i} (-1)^{\alpha_i} \frac{\mu^{(\alpha_i)}(H_i)}{\alpha_i!} (|t_i|^2)^{\alpha_i} \\
&= \prod_{i=1}^r \sum_{\alpha_i} (-\mu(H_i))_{(\alpha_i)} \frac{1}{\alpha_i!} (|t_i|^2)^{\alpha_i} \\
&= \prod_{i=1}^r {}_1F_0(-\mu(H_i); |t_i|^2)
\end{aligned}
$$

where $a_{(r)} = a(a+1)\cdots(a+r-1) = \frac{\Gamma(a+1)}{\Gamma(a+r-1)}$. By [6] the hypergeometric function ${}_1F_0(r;s) = (1-s)^{-r}$ converges whenever $|s| < 1$. It follows that the series for $q_c w$ converges whenever $0 \leq t_j < 1$. Thus $q_T v \in \mathbb{H}$, for each $v \in H$ and $T \in \mathcal{D}$.

To complete the proof it suffices to show $c \in \mathcal{D}$ when $q_c w$ converges where $c \in a_+$ and w is a nonzero weight vector in H. Indeed, note if $q_T w$ converges for some nonzero vector $w \in H$ and $T \in \mathfrak{p}_+$, then $\omega(k) q_T w = \omega(k) q_T \omega(k^{-1}) \tau(k) w = q_{k \cdot T} \tau(k) w$ converges and hence we may assume $T = c$ where $c \in a_+$. Next note if $q_c w$ converges for some nonzero $w \in H$, then $w = \sum w_i$ where w_i are nonzero weight vectors and $||q_c w||^2 = \sum_i ||q_c w_i||^2$. Hence $q_c w$ converges for some nonzero weight vector w having weight μ. Since the radius of convergence of ${}_1F_0(-\mu(H_i), |t_i|^2) = (1-|t_i|^2)^{\mu(H_i)}$ is 1 for $\mu(H_i) < 0$, equation (5.2) implies $c \in \mathcal{D}$ if all the $\mu(H_i)$ are negative.

Let $(\cdot,\cdot)$ denote the inner product on the dual $\mathfrak{t}^*$ of $\mathfrak{t}$ induced by the Killing form. For α and β in $\mathfrak{t}^*$ let $\alpha_\beta = \frac{2(\alpha,\beta)}{(\beta,\beta)}$. Our choices are such that $\mu(H_i) = \mu_{\gamma_i}$. First consider the case where $\mu = \lambda$ is the highest weight for (τ, H). Let ζ be defined as in equation (2.2) of [5]. Let $z \in \mathbb{R}$ and $\lambda_\circ$ be determined by the conditions $\lambda = \lambda_\circ + z\zeta$ and $(\lambda_\circ + \rho, \gamma_1) = 0$ where ρ is half the sum of the positive roots (cf. equation (2.3) of [5]). The condition $(\lambda, \gamma_1) < 0$ is equivalent to $z < \rho_{\gamma_1}$. However, since ω is a unitary highest weight representation, z is bounded from above by the number $B(\lambda_\circ)$ given by Theorem 2.4 and Theorem 2.8 of [5]. The explicit formula for $B(\lambda_\circ)$ involves weights $\rho_{c,1}$ and $\rho_{c,2}$ which are half the sums of the positive compact roots in certain distinguished subroot

systems Q and R of Φ. If ρ_c denotes half the sum of the compact roots of Φ then $\rho_c - \rho_{c,i}$ is half the sum of some positive compact roots for $i = 1, 2$. By maximality of the root γ_1 we have $(\alpha, \gamma_1) \geq 0$, whenever $\alpha \in \Phi_c^+$. From this it follows that $(\rho_{c,i})_{\gamma_1} \leq (\rho_c)_{\gamma_1}$ for each $i = 1, 2$. Thus by Theorem 2.8 of [5] $B(\lambda_\circ) \leq 1 + 2(\rho_c)_{\gamma_1}$. However, Lemma 5.3 of [5] gives $z < 1 + 2(\rho_c)_{\gamma_1} = \rho_{\gamma_1}$ in the case when $\lambda \neq 0$. Hence $\lambda_{\gamma_1} < 0$. Now suppose μ is an arbitrary weight in (τ, H). Then $\lambda - \mu$ is a sum of positive compact roots. Again by maximality of γ_1 we have $\mu_{\gamma_1} \leq \lambda_{\gamma_1} < 0$.

The strongly orthogonal roots all have the same length, and γ_1 is the highest weight of the irreducible representation of $\mathfrak{k}$ on $\mathfrak{p}_+$. Hence for each i there is a compact Weyl group element σ such that $\sigma\gamma_1 = \gamma_i$. Since the σ permutes the weights of (τ, H), it follows that $\mu_{\gamma_i} = \mu_{\sigma\gamma_1} = (\sigma^{-1}\mu)_{\gamma_1} < 0$. This completes the proof. □

6. The operator ω in terms of q_T

In this section we relate the operations of the group representation ω to the operator q_T. More precisely, we show

$$\omega(g)v = q_{g\cdot 0} J(g,0)^{*-1} v \text{ for } v \in H \text{ and } g \in G. \tag{6.1}$$

First we relate q_E to the representation ρ of a real Lie group G having a three-dimensional system $\{H, E, \bar{E}\}$ with highest weight vector v.

REMARK 6.1. *Suppose $\{H, E, \bar{E}\}$ forms a three-dimensional system in the complex Lie algebra of a complex Lie group $G_{\mathbb{C}}$. Then as seen by Lemma 7.7 of [7]*

$$\exp(t(E + \bar{E})) = \exp(\tanh t\bar{E}) \exp(\log(\cosh t)H) \exp(\tanh tE). \tag{6.2}$$

Moreover, if v is a highest weight vector with weight μ for the three-dimensional system $\{H, E, \bar{E}\}$ then by Theorem 5.1,

$$q_{\tanh t\, E} v = \sum \frac{\tanh^n t}{n!} \bar{E}^n v$$

is a convergent series in the Hilbert space and gives an analytic function of t. This suggests that

$$\rho(\exp(t(E + \bar{E}))v = \cosh^\mu t \sum \frac{\tanh^n t}{n!} \bar{E}^n v.$$

DEFINITION. *Suppose $\{H, E, \bar{E}\}$ forms a three-dimensional system and $Hv = \mu v$, $Ev = 0$. Define $c_{n,k}(\mu)$ by $(E + \bar{E})^n v = \sum_{k=0}^n c_{n,k}(\mu)\bar{E}^k v$.*

REMARK 6.2. *We have* $c_{0,0} = 1$, $c_{1,0} = 0$, $c_{1,1} = 1$. *Moreover*

$$\begin{aligned}(E+\bar{E})^2 v &= E^2 v + E\bar{E}v + \bar{E}Ev + \bar{E}^2 v \\ &= Hv + \bar{E}^2 v \\ &= \mu v + \bar{E}^2 v\end{aligned}$$

gives

$$\begin{aligned}&c_{2,0} = \mu \\ &c_{2,1} = 0 \text{ and} \\ &c_{2,2} = 1.\end{aligned} \tag{6.3}$$

LEMMA 6.1. *One has*

$$\begin{aligned}c_{n+1,0}(\mu) &= \mu c_{n,1}(\mu) \\ c_{n+1,k}(\mu) &= c_{n,k-1}(\mu) + (k+1)(\mu - k)c_{n,k+1}(\mu), \quad 0 < k < n \\ c_{n+1,n}(\mu) &= c_{n,n-1}(\mu) \\ c_{n+1,n+1}(\mu) &= c_{n,n}(\mu) = 1.\end{aligned}$$

PROOF. By Lemma 4.2

$$\begin{aligned}(E+\bar{E})^{n+1} v &= (E+\bar{E}) \sum_{k=0}^{n} c_{n,k} \bar{E}^k v \\ &= \sum_{k=0}^{n} c_{n,k} E \bar{E}^k v + \sum_{k=0}^{n} c_{n,k} \bar{E}^{k+1} v \\ &= \sum_{k=1}^{n} c_{n,k} k(\mu - k + 1) \bar{E}^{k-1} v + \sum_{k=0}^{n} c_{n,k} \bar{E}^{k+1} v \\ &= \sum_{k=0}^{n-1} c_{n,k+1}(k+1)(\mu - k) \bar{E}^k v + \sum_{k=1}^{n+1} c_{n,k-1} \bar{E}^k v \\ &= c_{n,1} \mu v + \sum_{k=1}^{n-1} \{(k+1)(\mu - k)c_{n,k+1} + c_{n,k-1}\} \bar{E}^k v \\ &\quad + c_{n,n-1} \bar{E}^n v + c_{n,n} \bar{E}^{n+1} v.\end{aligned}$$

Thus

$$\begin{aligned}c_{n+1,0} &= \mu c_{n,1} \\ c_{n+1,k} &= c_{n,k-1} + (k+1)(\mu - k)c_{n,k+1} \\ c_{n+1,n} &= c_{n,n-1} \\ c_{n+1,n+1} &= c_{n,n}.\end{aligned}$$

□

PROPOSITION 6.1. *Let v be a highest weight vector with weight μ for a three-dimensional system $\{H, E, \bar{E}\}$ relative to a representation ρ of a real Lie group G. Then for $t \geq 0$*

$$\begin{aligned}\rho(\exp(t(E+\bar{E}))v &= (\cosh t)^{\mu} \sum_{n=0}^{\infty} \frac{\tanh^n t}{n!} \bar{E}^n v \\ &= (\cosh t)^{\mu} q_{\tanh t\, E} v.\end{aligned}$$

PROOF. We will show

$$\sum_n \frac{t^n (E+\bar{E})^n}{n!} v = (\cosh t)^{\mu} \sum_n \frac{\tanh^n t}{n!} \bar{E}^n v \text{ for } t > 0.$$

This will occur if

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n} c_{n,k} \bar{E}^k v = (\cosh t)^{\mu} \sum_{n=0}^{\infty} \frac{\tanh^n t}{n!} \bar{E}^n v.$$

Hence we need to show

$$\sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{c_{n,k} t^n}{n!} \bar{E}^k v = (\cosh t)^{\mu} \sum_{k=0}^{\infty} \frac{\tanh^k t}{k!} \bar{E}^k v.$$

Thus for each k it suffices to show

$$\sum_{n=k}^{\infty} \frac{c_{n,k} t^n}{n!} = (\cosh t)^{\mu} \frac{\tanh^k t}{k!}. \tag{6.4}$$

Set $x = \cosh t$. Then $e^{2t} - 2xe^t + 1 = 0$. This gives $t = \ln(x \pm \sqrt{x^2-1})$. However, since $t > 0$, one has the plus sign. Indeed, if $t = \ln(x - \sqrt{x^2-1})$, then $e^t = \frac{1}{x+\sqrt{x^2-1}} < 1$. Thus

$$t = \ln(x + \sqrt{x^2-1}).$$

We now define $C_{n,k}/n!$ to be the power series coefficients of $(\cosh t)^{\mu} \frac{\tanh^k t}{k!}$. Note if $x = \cosh t$, then $\tanh t = \frac{\sqrt{x^2-1}}{x}$. Set

$$f_k(x) = (\cosh t)^{\mu} \frac{\tanh^k t}{k!} = \frac{1}{k!} x^{\mu-k} (x^2-1)^{k/2}.$$

Then

$$f_k(x) = \sum_{n=k}^{\infty} \frac{C_{n,k}}{n!} \ln(x + \sqrt{x^2-1})^n.$$

We calculate $f_k'(x)$ two ways. First

$$\begin{aligned} f_k'(x) &= \frac{\mu-k}{k!}x^{\mu-k-1}(x^2-1)^{k/2} + \frac{1}{(k-1)!}x^{\mu-k+1}(x^2-1)^{(k-2)/2} \\ &= (k+1)(\mu-k)\frac{1}{\sqrt{x^2-1}}\frac{1}{(k+1)!}x^{\mu-(k+1)}(x^2-1)^{(k+1)/2} \\ &\quad + \frac{1}{\sqrt{x^2-1}}\frac{1}{(k-1)!}x^{\mu-(k-1)}(x^2-1)^{(k-1)/2} \\ &= \frac{1}{\sqrt{x^2-1}}\left\{(k+1)(\mu-k)f_{k+1}(x) + f_{k-1}(x)\right\}. \end{aligned}$$

We next calculate $f_k'(x)$ from the power series.

$$\begin{aligned} f_k'(x) &= \sum_{n=k}^{\infty}\frac{C_{n,k}}{(n-1)!}\left(\ln(x+\sqrt{x^2-1})\right)^{n-1}\frac{1}{x+\sqrt{x^2-1}}(1+\frac{x}{\sqrt{x^2-1}}) \\ &= \frac{1}{\sqrt{x^2-1}}\sum_{n=k}^{\infty}\frac{C_{n,k}}{(n-1)!}\left(\ln(x+\sqrt{x^2-1})\right)^{n-1}. \end{aligned}$$

Combining these we obtain

$$(k+1)(\mu-k)f_{k+1}(x) + f_{k-1}(x) = \sum_{n=k}^{\infty}\frac{C_{n,k}}{(n-1)!}\left(\ln(x+\sqrt{x^2-1})\right)^{n-1}.$$

These yield recursion formulas for the $C_{n,k}$. First we rewrite this as

$$(k+1)(\mu-k)f_{k+1}(x) + f_{k-1}(x) = \sum_{n=k-1}^{\infty}\frac{C_{n+1,k}}{n!}\left(\ln(x+\sqrt{x^2-1})\right)^{n}.$$

This gives

$$\begin{aligned} C_{k-1,k-1} &= C_{k,k} \\ C_{k,k-1} &= C_{k+1,k} \\ (k+1)(\mu-k)C_{n,k+1} + C_{n,k-1} &= C_{n+1,k} \text{ for } n \geq k+1. \end{aligned}$$

Now

$$\cosh^{\mu} t = \sum_{n=0}^{\infty}\frac{C_{n,0}}{n!}t^n = \sum_{n=0}^{\infty}\frac{f_0^{(n)}(0)}{n!}t^n.$$

Hence $C_{0,0} = 1$, $C_{1,0} = 0$, $C_{2,0} = \mu$. By Lemma 6.1 and equation 6.3, $C_{n,k}$ and $c_{n,k}$ satisfy identical recursion relations and agree for $k = 0$ and $n \in \{0, 1, 2\}$. Hence we have

$$C_{n,k} = c_{n,k}$$

for all n and k. □

We now establish the relation between the representation ω and the operator q_T. Let $Z \in \mathfrak{a}_0$ be written in the form

$$Z = \sum_{i=1}^{r} t_i(E_i + \bar{E}_i) \tag{6.5}$$

where $E_i = E_{\gamma_i}$ are the root vectors of the strongly orthogonal roots γ_i. Then by Lemma 7.11 of [7] one has the decomposition

$$\begin{aligned} \exp(Z) &= \exp(\bar{X})\exp(H)\exp(X) \\ &= \exp(X)\exp(-H)\exp(\bar{X}) \end{aligned} \tag{6.6}$$

where

$$H = \sum_{i=1}^{r} \log(\cosh t_i) H_i \text{ and } X = \sum_{i=1}^{r} \tanh t_i E_i.$$

Let $v \in H$ be a weight vector with weight μ. Recall that $E_i v = 0$ for each i. We set $\mu_i = \mu(H_i)$. Then $H_i \cdot v = \mu_i v$ for each i. Therefore v is a highest weight vector for the three-dimensional systems $\{H_i, E_i, \bar{E}_i\}$ relative to ω.

PROPOSITION 6.2. *One has*

$$\begin{aligned} \omega(\exp(\sum t_i(E_i + \bar{E}_i)))v &= \prod(\cosh t_i)^{\mu_i} \exp(\sum \tanh t_i \bar{E}_i)v \\ &= \prod(\cosh t_i)^{\mu_i} q_{\sum \tanh t_i \bar{E}_i} v. \end{aligned}$$

PROOF. The series on both sides converge. Hence we need only show they are equal. Consider the left-hand side

$$\exp(\sum t_i(E_i + \bar{E}_i))v = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{t^\alpha}{\alpha!}(E + \bar{E})^\alpha v$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_r)$ and $t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_r^{\alpha_r}$ and

$$(E + \bar{E})^\alpha = (E_1 + \bar{E}_1)^{\alpha_1} \cdots (E_r + \bar{E}_r)^{\alpha_r}.$$

Note

$$\begin{aligned} (E_i + \bar{E}_i)^{\alpha_i}(E_j + \bar{E}_j)^{\alpha_j} v &= (E_i + \bar{E}_i)^{\alpha_i} \left\{ \sum_{\beta_j=0}^{\alpha_j} c_{\alpha_j,\beta_j}(\mu_j) \bar{E}_j^{\beta_j} v \right\} \\ &= \sum_{\beta_j=0}^{\alpha_j} c_{\alpha_i,\beta_j}(\mu_j) \bar{E}_j^{\beta_j} (E_i + \bar{E}_i)^{\alpha_i} v \\ &= \sum_{\beta_i=0}^{\alpha_i} \sum_{\beta_j=0}^{\alpha_j} c_{\alpha_i,\beta_i}(\mu_i) c_{\alpha_j,\beta_j}(\mu_j) \bar{E}_i^{\beta_i} \bar{E}_j^{\beta_j} v. \end{aligned}$$

Hence define $c_{\alpha,\beta}(\mu) = \prod c_{\alpha_i,\beta_i}(\mu_i)$ where $\beta \leq \alpha$. Set $\bar{E}^\beta = \bar{E}_1^{\beta_1} \cdots \bar{E}_r^{\beta_r}$. We thus see

$$(E + \bar{E})^\alpha v = \sum_{\beta \leq \alpha} c_{\alpha,\beta}(\mu) \bar{E}^\beta v.$$

Hence

$$\exp(\sum t_i(E_i+\bar{E}_i))v = \sum_{k=0}^{\infty}\sum_{|\alpha|=k}\frac{1}{\alpha!}t^{\alpha}\sum_{\beta\le\alpha}c_{\alpha,\beta}(\mu)\bar{E}^{\beta}v$$
$$= \sum_{k=0}^{\infty}\sum_{|\beta|\le k}\sum_{\substack{\alpha\ge\beta\\|\alpha|=k}}\frac{1}{\alpha!}t^{\alpha}c_{\alpha,\beta}(\mu)\bar{E}^{\beta}v.$$

Now the right hand side can be rewritten in a similar way.

$$\prod(\cosh t_i)^{\mu_i}\exp(\sum\tanh t_i\bar{E}_i)v = \prod(\cosh t_i)^{\mu_i}\sum_{k=0}^{\infty}\frac{1}{k!}(\sum\tanh t_i\bar{E}_i)^k v$$
$$= \prod(\cosh t_i)^{\mu_i}\sum_{k=0}^{\infty}\sum_{|\beta|=k}\frac{1}{\beta!}(\tanh t)^{\beta}\bar{E}^{\beta}v$$

where $(\tanh t)^{\beta}\equiv\prod\tanh t_i^{\beta_i}$. Hence we have equality if their projections onto $\mathbb{H}_l$ are equal. Hence we need to show

$$\sum_{k=0}^{\infty}\sum_{|\beta|=l}\sum_{\substack{\alpha\ge\beta\\|\alpha|=k}}\frac{1}{\alpha!}t^{\alpha}c_{\alpha,\beta}(\mu)\bar{E}^{\beta}v = \prod(\cosh t_i)^{\mu_i}\sum_{|\beta|=l}\frac{1}{\beta!}(\tanh t)^{\beta}\bar{E}^{\beta}v.$$

But this follows if

$$\sum_{\alpha\ge\beta}\frac{c_{\alpha,\beta}(\mu)}{\alpha!}t^{\alpha} = \frac{1}{\beta!}\prod(\cosh t_i)^{\mu_i}(\tanh t)^{\beta}.$$

This however is implied by (6.4). □

THEOREM 6.1. *Let $v\in H$ and $g\in G$. Then*

$$\omega(g)v = q_{g\cdot 0}J(g,0)^{*-1}v.$$

PROOF. It is enough to assume that v is a weight vector in H with weight μ. To establish the formula for $a\in A$ it suffices by Proposition 6.2 to note that $a\cdot 0 = \pi_+(a) = \sum\tanh t_i E_i$ and $J(a,0)^{*-1}v = \prod(\cosh t_i)^{\mu_i}v$, when $a = \exp(\sum_{i=1}^{r}t_i(E_i+\bar{E}_i))$. By equation (6.6) we can write $a = \exp(X)\exp(-H)\exp(\bar{X})$, where $H = \sum_{i=1}^{r}\log(\cosh(t_i))H_i$ and $X = \sum_{i=1}^{r}\tanh t_i E_i$. Thus $a\cdot 0 = \sum_i\tanh t_i E_i$ and

$$J(a,0)^{*-1}v = \tau(\pi_{\circ}(a))^{*-1}v = \tau(\exp(-H))^{*-1}v = \tau(\exp(H))v = \prod(\cosh t_i)^{\mu_i}v.$$

For general $g \in G$, we can write $g = k_1 a k_2$ where $k_1, k_2 \in K$. Then

$$\begin{aligned}
\omega(g)v &= \omega(k_1)\omega(a)\omega(k_2)v \\
&= \omega(k_1)\omega(a)\tau(k_2)v \\
&= \omega(k_1)q_{a\cdot 0}J(a,0)^{*-1}\tau(k_2)v \\
&= \omega(k_1)q_{ak_2\cdot 0}\omega(k_1)^{-1}\tau(k_1)J(a,k_20)^{*-1}J(k_2,0)^{*-1}v \\
&= q_{\mathrm{Ad}(k_1)(ak_2\cdot 0)}J(k_1, ak_2\cdot 0)^{*-1}J(ak_2,0)^{*-1}v \\
&= q_{k_1ak_2\cdot 0}J(k_1ak_2,0)^{*-1}v \\
&= q_{g\cdot 0}J(g,0)^{*-1}v.
\end{aligned}$$

□

7. Intertwining Operators and Reproducing Kernels

The operator q_T will play a decisive role in determining the intertwining operator between the given realization $(\omega, \mathbb{H})$, and the standard realization of the highest weight representation as a space of H-valued functions on $\mathcal{D}$. Furthermore, q_T will be used to determine the reproducing kernel for the standard realization.

PROPOSITION 7.1. *Let $g \in G$, $T \in \mathcal{D}$ and $v \in H$. Then*

$$\omega(g)q_T v = q_{g\cdot T}J(g,T)^{*-1}v.$$

PROOF. Choose g_1 so that $g_1 \cdot 0 = T$. Then by Theorem 6.1

$$\begin{aligned}
\omega(g)q_T v &= \omega(g)\omega(g_1)J(g_1,0)^* v \\
&= \omega(gg_1)J(g_1,0)^* v \\
&= q_{gg_1\cdot 0}J(gg_1,0)^{*-1}J(g_1,0)^* v \\
&= q_{g\cdot T}J(g, g_1\cdot 0)^{*-1}J(g_1,0)^{*-1}J(g_1,0)^* v \\
&= q_{g\cdot T}J(g,T)^{*-1}v.
\end{aligned}$$

□

PROPOSITION 7.2. *The span of the vectors $q_T v$, $T \in \mathcal{D}$, $v \in H$ is dense in the Hilbert space $\mathbb{H}$.*

PROOF. This follows from Proposition 7.1 and the irreducibility of ω. □

LEMMA 7.1. *For vectors $v, w \in H$, we have $\langle q_T v | w\rangle = \langle v | w\rangle$.*

PROOF. Let $T \in \mathcal{D}$ and $v \in H$. Then by equation (3.2) $q_T v = \exp(\bar{T})v = v + v'$, where $v' \in \oplus_{i\geq 1}\mathbb{H}_i$. By Remark 3.1 v' is orthogonal to $H = \mathbb{H}_0$. The result now follows. □

THEOREM 7.1. *There is a positive definite operator-valued function* $Q : \mathcal{D} \times \mathcal{D} \to Aut(H_\tau)$ *such that*

$$\langle q_T v | q_S w \rangle = \langle Q(S,T)v | w \rangle.$$

The function Q *is holomorphic in the variable* S *and antiholomorphic in the variable* T.

PROOF. Choose g_1 and g_2 so that $g_1 \cdot 0 = T$ and $g_2 \cdot 0 = S$. Then Proposition 7.1 gives

$$q_T v = \omega(g_1) J(g_1, 0)^* v \text{ and}$$
$$q_S w = \omega(g_2) J(g_2, 0)^* w.$$

Then

$$\begin{aligned}
\langle q_T v | q_S w \rangle &= \langle \omega(g_2^{-1} g_1) J(g_1, 0)^* v | J(g_2, 0)^* w \rangle \\
&= \langle q_{g_2^{-1} g_1 \cdot 0} J(g_2^{-1} g_1, 0)^{*-1} J(g_1, 0)^* v | J(g_2, 0)^* w \rangle \\
&= \langle J(g_2^{-1}, g_1 \cdot 0)^{*-1} v | J(g_2, 0)^* w \rangle \text{ by Lemma 7.1} \\
&= \langle J(g_2, 0) J(g_2^{-1}, T)^{*-1} v | w \rangle.
\end{aligned}$$

Now write $g_2 = \pi_+(g_2)\pi_\circ(g_2)\pi_-(g_2)$. In the following let $\sigma : G_{\mathbb{C}} \to G_{\mathbb{C}}$ denote the real Lie group homomorphism whose differential is conjugation on $\mathfrak{g}$. Then we have

$$g_2^{-1} = \sigma(g_2)^{-1} = \sigma\pi_-(g_2)^{-1}\sigma\pi_\circ(g_2)^{-1}\sigma\pi_+(g_2)^{-1}.$$

From (2.6) and the definition of $J(g,T)$, we easily obtain

$$J(xg, T) = J(g, T) \text{ if } x \in P^+ \tag{7.1}$$

$$J(kg, T) = \tau(k) J(g, T) \text{ if } k \in K_{\mathbb{C}}. \tag{7.2}$$

From this we get

$$\begin{aligned}
J(g_2, 0) J(g_2^{-1}, T)^{*-1} &= \tau(\pi_\circ(g_2)) J(\sigma\pi_-(g_2)^{-1}\sigma\pi_\circ(g_2)^{-1}\sigma\pi_+(g_2)^{-1}, T)^{*-1} \\
&= \tau(\pi_\circ(g_2)) J(\sigma\pi_-(g_2)^{-1}\sigma\pi_\circ(g_2)^{-1}, \sigma\pi_+(g_2)^{-1} \cdot T)^{*-1} J(\sigma\pi_+(g_2)^{-1}, T)^{*-1} \\
&= \tau(\pi_\circ(g_2)) \tau(\sigma\pi_\circ(g_2)^{-1})^{*-1} J(\sigma\pi_+(g_2)^{-1}, T)^{*-1} \\
&= J(\exp(-\bar{S}), T)^{*-1}.
\end{aligned}$$

It follows now that $Q(S,T) = J(\exp(-\bar{S}), T)^{*-1}$ and $Q(S,T)$ is holomorphic in S and conjugate holomorphic in T. □

PROPOSITION 7.3. *The reproducing kernel* Q *is positive definite.*

PROOF. By Proposition 7.1

$$\sum_{i,j} \langle Q(T_i, T_j) v_j | v_i \rangle = \langle \sum q_{T_i} v_i | \sum q_{T_i} v_i \rangle \geq 0.$$

□

Since $Q(\cdot,\cdot)$ is a positive definite operator valued kernel we can follow Kunze [8] to construct a Hilbert space $\mathcal{H}(\mathcal{D},\tau)$ of continuous functions $f:\mathcal{D}\to H$ with the following properties:

(1) The span of the set $\{S\mapsto Q(S,T)v\,:\,T\in D,\,v\in H\}$ is dense in $\mathcal{H}(\mathcal{D},\tau)$;
(2) For $S\in\mathcal{D}$, the map $e_S:f\to f(S)$ is a continuous map from $\mathcal{H}(\mathcal{D},\tau)$ to H;
(3) $Q(S,T)=e_Se_T^*$, for all $S,T\in\mathcal{D}$;
(4) $(Q(\cdot,T)v|Q(\cdot,S)w)=(Q(S,T)v|w)$, for all $S,T\in\mathcal{D}$ and $v,w,\in H$.

When τ is singular, the space $\mathcal{H}(\mathcal{D},\tau)$ is a proper subspace of $\mathrm{Hol}(\mathcal{D},\tau)$ and can be described as the solutions to certain differential operators. We refer the reader to [3, 4] for more details.

PROPOSITION 7.4. *The formula*

$$\mathcal{T}(g)f(S)=J(g^{-1},S)^{-1}f(g^{-1}\cdot S),$$

where $f\in\mathcal{H}(\mathcal{D},\tau)$, defines a strongly continuous unitary representation of G on $\mathcal{H}(\mathcal{D},\tau)$.

PROOF. This is easily verified. See [8] for details. □

PROPOSITION 7.5. *Let $S,T\in\mathcal{D}$ and let $g\in G$. Then*

$$Q(g\cdot S,g\cdot T)=J(g,S)\,Q(S,T)\,J(g,T)^*.$$

PROOF. By the definition of $\mathcal{T}$ one can easily verify that $e_{g\cdot T}=J(g,T)\,e_T\,\mathcal{T}(g^{-1})$. From this we get

$$\begin{aligned}Q(g\cdot S,g\cdot T)=&e_{g\cdot S}\,e_{g\cdot T}^*\\=&J(g,S)\,e_S\,\mathcal{T}(g^{-1})\mathcal{T}(g^{-1})^*\,e_T^*\,J(g,T)^*\\=&J(g,S)\,e_Se_T^*\,J(g,T)^*\\=&J(g,S)\,Q(S,T)\,J(g,T)^*.\end{aligned}$$

□

THEOREM 7.2. *The mapping $q_Tv\mapsto Q(\cdot,T)v$ extends to an intertwining operator Ξ between $(\omega,\mathbb{H})$ and the geometric representation $(T,\mathcal{H}(\mathcal{D},H))$. It can be defined globally by $\Xi f(S)=q_S^*f$ for $f\in\mathbb{H}$.*

PROOF. Let $v_i\in V$ and $T_i\in\mathcal{D}$. By the definition of the inner product on $\mathcal{H}(\mathcal{D},\tau)$ and by Theorem 7.1 we have

$$\begin{aligned}\|\sum_i Q(\cdot,T)v_i\|^2=&\sum_{i,j}\langle Q(T,T)v_i|v_j\rangle\\=&\sum_{i,j}\langle q_{T_i}v_i|q_{T_j}v_j\rangle\\=&\|\sum_i q_{T_i}v_i\|^2.\end{aligned}$$

By Proposition 7.2 it follows that the map $q_T v \to Q(\cdot, T)v$ is well-defined and extends to a unitary map $\Xi : \mathbb{H} \to \mathcal{H}(\mathcal{D}, \tau)$. Let $g \in G$ and let $T \in \mathcal{D}$. Then we have by Propositions 7.1 and 7.5

$$\begin{aligned}\Xi(\omega(g)q_T v) =& \Xi(q_{gT} J(g,T)^{*-1} v) \\ =& Q(\cdot, gT) J(g,T)^{*-1} v \\ =& J(g^{-1}, \cdot) Q(g^{-1}(\cdot), T) v \\ =& \mathcal{T}(g)(Q(\cdot, T)v) \\ =& \mathcal{T}(g)\Xi(q_T v).\end{aligned}$$

It now follows that Ξ is a unitary operator intertwining $(\omega, \mathbb{H})$ and $(\mathcal{T}, \mathcal{H}(\mathcal{D}, \tau))$. □

8. An Example

As an example and application we consider the harmonic representation ω. Discussions of this representation appear in many articles in the literature. We will adopt much of the notation and results stated in Chapter 7 of [4]. Let $G = SU(p,q)$, $Sp(n, \mathbb{R})$, or $SO^*(2m)$. The Hilbert space for the action of G is described as follows: We let M be the complex vector space which depends on G and is described by (7.2) of [4]. Then the Hilbert space is the Fock space $\mathbb{F}$ consisting of all those holomorphic functions on M which are square integrable with respect to the Gaussian measure. Let $\theta : M \to \mathfrak{p}_+$ be the map defined there and $\mathcal{I}$ the ideal generated by the matrix entries of θ and the constants. If H is an irreducible K-space of harmonic polynomials then the closure $\mathbb{H}$ of $\mathcal{I}H$ is an irreducible subspace of $\mathbb{F}$ for ω. On $\mathfrak{p}_+$ we let $(\cdot|\cdot)$ denote the standard complex inner product induced by the Killing form. Then for $x \in \mathfrak{p}_+$ the action of $\bar{x}$ is given by $\bar{x} \cdot f(z) = (\theta(z)|x) f(z)$. From this it follows that for $T \in \mathcal{D}$ the operator q_T is the multiplication operator

$$q_T h(z) = e^{(\theta(z)|T)} h(z) \tag{8.1}$$

where $h \in H$. The intertwining operator is determined by the formula

$$\langle \Xi\phi(T) | h \rangle = \langle \phi | q_T h \rangle, \tag{8.2}$$

where $\phi \in \mathbb{H}$, $h \in H$, and $T \in \mathcal{D}$. If $K(\cdot, \cdot)$ is the reproducing kernel on H then the intertwining operator is given explicitly by the formula:

$$\Xi\phi(T, z) = \int \phi(w) e^{(T|\theta(w))} \overline{K(w, z)} d\mu(w). \tag{8.3}$$

References

1. Jean-Louis Clerc, *Equivariant holomorphic map into the Siegel disk and the metaplectic representation*, preprint, University of Nancy.
2. M. G. Davidson, *The harmonic representation of U(p,q) and its connection with the generalized unit disk*, Pacific J. Math. **129** (1987), 33 – 55.

3. M. G. Davidson, T. J. Enright and R. Stanke, *Covariant differential operators*, Math. Ann. **288** (1990), 731–739.
4. ———, *Differential Operators and Highest Weight Representations*, Memoirs of the Amer. Math. Soc. **94** (1991).
5. T. J. Enright, R. Howe and N. Wallach, *A classification of unitary highest weight modules*, In: Proceedings, Representation Theory of Reductive Groups (Park City, UT, 1982), 97–143; *Progr. Math.* **40** (1983), 97–143.
6. I. Gradshteyn and I. Ryzhik, *Tables of Integrals, Series, and Products*, Academic Press, London, 1980.
7. S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York, 1978.
8. R. A. Kunze, *Positive definite operator-valued kernels and unitary representations*, In: Proceedings of the Conference on Functional Analysis at Irvine, California, (1966), 235–247, Thompson Book Company.
9. L. A. Mantini, *An integral transform in L^2-cohomology for the ladder representations of $U(p,q)$*, J. Funct. Anal. **60** (1985), 211–242.
10. ———, *An L^2-cohomology construction of negative spin mass zero equations for $U(p,q)$*, J. Math. Anal. and Appl. **136** (1988), 419–449.

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Hankel Transforms and Holomorphic Discrete Series

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Dedicated to R. A. Kunze, on the occasion of his sixty-sixth birthday

ABSTRACT. This paper computes the Hankel transform $\mathcal{K}_\pi$ of the automorphism group G of a Siegel domain of type II explicitly. Using this new description, we study the holomorphic discrete series of G.

In [1], we studied automorphism groups G of Siegel domains Σ and their representations by the Jordan pair structure. Essentially, the special element j of G is the symmetry of Σ with respect to the point ie. Realized in a vector-valued Schrödinger-Fock Hilbert space $\mathfrak{SF}_\Gamma$, the discrete series S_π of G at j is the Hankel transform $\mathcal{K}_\pi$. In [1], we defined $\mathcal{K}_\pi$ as an integral operator, the kernel of which is the Bessel function K_π, and we showed that the square-integrability of K_π implies the square-integrability of the discrete series of G. This approach generates the theory developed in [6], [7], [2]. In this paper, we compute $\mathcal{K}_\pi$ explicitly on a dense subspace of $\mathfrak{SF}_\Gamma$. Using this new description, we prove the square-integrability of S_π. This new approach to studying the discrete series of G is independent of Bessel functions.

This paper is dedicated to Ray A. Kunze on the occasion of his sixty-sixth birthday. I am deeply appreciative of the contributions he has made to my mathematical career.

We now describe our new approach in detail. We review the necessary background materials first (cf. [1] also), then turn to the main results of this paper, which are contained in three theorems.

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Let (V, V^-) be a simple Jordan pair with a positive Hermitian involution $x \mapsto \bar{x}$. With respect to a maximal tripotent e, V is decomposed as $V = V_2 \oplus V_1$. Then V_2 is the complexification of a formally real Jordan algebra J, in which the interior Ω of the set of squares is the positive cone. With a positive definite bilinear map $B : V_1 \times V_1^- \to V_2$ we can define a Siegel domain Σ of type II by

$$\Sigma = \{(z, u) \in V_2 \oplus V_1 | Im\, z - \frac{1}{2} B(u, \bar{u}) > 0\}. \tag{1}$$

Let Δ be the Koecher norm function on Ω, r be the rank of Ω, n_1 and n be the dimensions of V_2 and V over $\mathbb{C}$, respectively. Then $n = n_1 + rb$ for some positive integer b. Denote $\mu = \frac{n_1}{r}$, which is an integer or a half-integer. The Cayley transform c is a biholomorphic isomorphism of a circled bounded symmetric domain D with Σ. As in [1], we define a trilinear map $\{\ \} : V \times V^- \times V \to V$ and define

$$x \circ y = \{x\bar{e}y\} \tag{2}$$

where $x, y \in V$.

Let G' and G be the groups of biholomorphic automorphisms of D and Σ, respectively, K be the maximal compact subgroup of G', and $K^{\mathbb{C}}$ the complexification of K. As in [3], we decompose the Lie algebra $\mathfrak{g}$ of G as a canonical graduation

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-\frac{1}{2}} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\frac{1}{2}} \oplus \mathfrak{g}_1, \tag{3}$$

and denote by G_{-1}, $G_{-\frac{1}{2}}$, G_0, $G_{\frac{1}{2}}$, and G_1 the corresponding analytic subgroups.

Viewing G as an automorphism group of the domain Σ given by (1), G_0, $G_{\frac{1}{2}}$, and G_1 act on Σ as follows.

(a) Elements a in G_0 can be described as

$$a \cdot (z, u) = (az, au). \tag{4}$$

I.e., as an operator on V_2, $a \in GL(V_2)$ and $a\Omega = \Omega$; as an operator on V_1, $a \in GL(V_1)$. Moreover,

$$a(B(u, \bar{u})) = B(au, \overline{au}). \tag{5}$$

(b) Elements in $G_{\frac{1}{2}}$ can be parametrized by elements u_0 in V_1; i.e., for $u_0 \in V_1$

$$R(u_0) \cdot (z, u) = (z + iB(u, \bar{u}_0) + \frac{i}{2} B(u_0, \bar{u}_0), u + u_0). \tag{6}$$

(c) Elements in G_1 can be parametrized by elements x_0 in J; i.e., for $x_0 \in J$

$$R(x_0) \cdot (z, u) = (z + x_0, u). \tag{7}$$

We also study the element j given by

$$j \cdot (z, u) = (-z^{-1}, -z^{-1} \circ u). \tag{8}$$

Let A be the connected component of the identity in G_0, and L the maximal compact subgroup of A. We identify V_1 with a subgroup $K_{-\frac{1}{2}}$ of $K^{\mathbb{C}}$ by the map $u_0 \mapsto k(u_0)$, where $k(u_0)$ acts on $V_2 \oplus V_1$ by $k(u_0)\cdot(z,u) = (z, z\circ u_0 + u)$. Note that as subgroups of $K^{\mathbb{C}}$, A normalizes $K_{-\frac{1}{2}}$, and $AK_{-\frac{1}{2}}$ is a subgroup of $K^{\mathbb{C}}$.

Let π be a finite-dimensional irreducible holomorphic representation of $K^{\mathbb{C}}$, denote by $\mathcal{V} = \mathcal{V}_\pi$ the space on which π acts and by $\mathcal{E} = \mathcal{E}_\pi$ the space of all linear transformations on $\mathcal{V}$. Let $M = (m_1, m_2, \cdots, m_r)$ with $m_1 \geq m_2 \geq \cdots \geq m_r$ be the signature of the restricted lowest weight of π, and set $\omega(\pi) = m_r$. We will study $\mathcal{V}$-valued or $\mathcal{E}$-valued measurable functions f on $AK_{-\frac{1}{2}}$, which are holomorphic in $u_0 \in V_1$. f is called π-covariant if

$$f(a\,k(u_0)\,l) = \pi(l)^{-1} f(a\,k(u_0)) \tag{9}$$

for all $a \in A$, $u_0 \in V_1$, and $l \in L$.

We recall from [1] that the $\mathcal{E}$-valued gamma function Γ on Σ is defined by

$$\begin{aligned}\Gamma((z,u),\alpha,\pi) = \int_{AK_{-\frac{1}{2}}} & e^{2i(az|e)}\, e^{2(au|u_0)}\, e^{-(u_0|u_0)} \\ & \times \pi((ak(iu_0))^*\, ak(iu_0))^{-1}\, \Delta(ae)^\alpha\, da\, du_0\end{aligned} \tag{10}$$

for $\alpha \in \mathbb{C}$ and $(z,u) \in \Sigma$. When $\omega(\pi) > \mu - Re\,\alpha - 1$, the integral (10) defining the gamma function converges absolutely. We set

$$\Gamma_\pi(\alpha) = \Gamma((\frac{ie}{2}, 0), \alpha, \pi). \tag{11}$$

In this paper, we assume

$$\omega(\pi) > 2\mu + b - 1. \tag{12}$$

Then

$$\Gamma_\pi = \Gamma_\pi(-\mu - b) \tag{13}$$

is defined.

Let $\mathfrak{GF}_\pi(AK_{-\frac{1}{2}})$ be the space of all $\mathcal{V}$-valued π-covariant measurable functions f on $AK_{-\frac{1}{2}}$ which are holomorphic in u_0 and such that

$$\int_{AK_{-\frac{1}{2}}} ||f(a\,k(u_0))||^2\, e^{-(u_0|u_0)}\, da\, du_0 < \infty. \tag{14}$$

$\mathfrak{GF}_\pi(AK_{-\frac{1}{2}})$ is a Hilbert space with respect to an inner product,

$$\langle f_1 | f_2 \rangle = \int_{AK_{-\frac{1}{2}}} (\Gamma_\pi f_1(ak(u_0)) | f_2(ak(u_0)))\, e^{-(u_0|u_0)}\, da\, du_0 \tag{15}$$

and we denote the resulting Hilbert space by $\mathfrak{GF}_\Gamma$.

For any $f \in \mathfrak{SF}_\Gamma$,

$$(16)\quad F(z,u) = \pi^{-\frac{n}{2}} \int_{AK_{-\frac{1}{2}}} e^{i(z|ae)}\, e^{(u|au_0)}\, e^{-(u_0|u_0)}\, \pi(ak(iu_0))^{*^{-1}}\, f(ak(u_0))\, da\, du_0$$

is absolutely convergent for all $(z,u) \in \Sigma$ and defines a holomorphic function $F : \Sigma \to \mathcal{V}_\pi$. We think of F as the Laplace transform of f. Denote by $\mathcal{H}_\pi$ the space of all functions F on Σ of the form (16) for $f \in \mathfrak{SF}_\pi$. Let P be the quadratic representation of the Jordan algebra $V_2 = J^{\mathbb{C}}$, and for $z_0 \in V_2$ define a mapping $r(z_0)$ on $V_2 \oplus V_1$ by

$$r(z_0) : (z,u) \mapsto (P(z_0)z, z_0 \circ u).$$

Then $\mathcal{H}_\pi$ is a Hilbert space with the inner product given by

$$(17)\quad \langle F|G\rangle = \int_\Sigma \operatorname{tr}\Big\{ G(x+iy,u)^*\, \pi(k(iu))\, \pi(r(2y - B(u,\bar{u})))^{-1} \times \pi(k(iu))^*\, F(x+iy,u)\Big\} \Delta(2y - B(u,\bar{u}))^{-2\mu - b}\, dx\, dy\, du,$$

and the mapping $\mathcal{T} : f \mapsto F$ given by (16) is a unitary map of $\mathfrak{SF}_\Gamma$ onto $\mathcal{H}_\pi$. Moreover, $\mathcal{H}_\pi$ is the space of all holomorphic functions $F : \Sigma \to \mathcal{V}_\pi$ satisfying

$$\int_\Sigma \|\pi(r(2y - B(u,\bar{u}))^{\frac{1}{2}}\, k(-iu))^{*^{-1}}\, F(x+iy,u)\|^2 \times \Delta(2y - B(u,\bar{u}))^{-2\mu - b}\, dx\, dy\, du < \infty.$$

Elements a in G_0, $R(u_0)$ in $G_{\frac{1}{2}}$, $R(x_0)$ in G_1, and the element j generate the group G. For these generators the unitary representation T_π of G acts on $\mathcal{H}_\pi$ as follows

$$(18)\quad \begin{cases} (T_\pi(a)\, F)(z,u) = \pi(a)\, F(a^{-1}z, a^{-1}u), \\ (T_\pi(R(u_0))\, F)(z,u) = \pi(k(-i\,u_0))^*\, F(z - i\,B(u,\bar{u}_0) \\ \qquad\qquad + \tfrac{i}{2}\, B(u_0,\bar{u}_0), u - u_0), \\ (T_\pi(R(x_0))\, F)(z,u) = F(z - x_0, u), \\ (T_\pi(j)\, F)(z,u) = \pi(r(z))\pi(k(u))\, F(-z^{-1}, z^{-1} \circ u). \end{cases}$$

By (16) and (18), $S_\pi = \mathcal{T}^{-1} T_\pi \mathcal{T}$ is a unitary representation of G on $\mathfrak{SF}_\Gamma$ given by

$$(19)\quad \begin{cases} (S_\pi(a_0)\, f)(a\,k(u)) = f(a_0^*\, a\, k(u)), \\ (S_\pi(R(u_0))\, f)(a\,k(u)) = e^{-\frac{1}{2}(a^*\,u_0|a^*\,u_0)}\, e^{(u|a^*\,u_0)}\, f(a\,k(u - a^*\,u_0)), \\ (S_\pi(R(x_0))\, f)(a\,k(u)) = e^{-i(x_0|a\,e)}\, f(a\,k(u)), \\ (S_\pi(j)\, f)(a\,k(u)) = (\mathcal{K}_\pi f)(a\,k(u)), \end{cases}$$

where $\mathcal{K}_\pi = \mathcal{T}^{-1} T_\pi(j) \mathcal{T}$ is called the Hankel transform on $\mathfrak{SF}_\Gamma$. In [1], $\mathcal{K}_\pi$ is an integral operator, the kernel of which is the Bessel function K_π on $AK_{-\frac{1}{2}}$. We now compute $\mathcal{K}_\pi$ explicitly on a dense subspace of $\mathfrak{SF}_\Gamma$.

For $(z', u') \in \Sigma$, we define an $\mathcal{E}$-valued function $f_{(z',u')}(\cdot)$ by

$$f_{(z',u')}(ak(u_0)) = \pi^{-\frac{n}{2}}\, e^{-i(ae|z')}\, e^{(au_0|u')}\, \Gamma_\pi^{-1}\, \pi(ak(iu_0))^{-1}. \tag{20}$$

It is clear that $f_{(z',u')}(\cdot)v$ satisfies (9). By (10) and (12), $f_{(z',u')}(\cdot)v$ also satisfies (14). Thus, $f_{(z',u')}(\cdot)v \in \mathfrak{SF}_\Gamma$ for every $v \in \mathcal{V}$.

THEOREM 1. *The functions $f_{(z',u')}(\cdot)v$ for $(z', u') \in \Sigma$ and $v \in \mathcal{V}$ span a dense subspace of $\mathfrak{SF}_\Gamma$.*

PROOF. By (15), (16) and (20), for $F = \mathcal{T}f$ and $v \in \mathcal{V}$ we have

$$(F(z', u')|v) = \langle f | f_{(z',u')}(\cdot)v \rangle.$$

Since $\mathcal{T}$ is a unitary map of $\mathfrak{SF}_\Gamma$ onto $\mathcal{H}_\pi$,

$$(F(z', u')|v) = \langle F | F_{(z',u')}(\cdot)v \rangle, \tag{21}$$

where $F_{(z',u')} = \mathcal{T} f_{(z',u')}$; i.e., $F_{(z',u')}$ is the reproducing kernel of $\mathcal{H}_\pi$. If $F \in \mathcal{H}_\pi$ is orthogonal to $F_{(z',u')}(\cdot)v$ for all $(z', u') \in \Sigma$ and $v \in \mathcal{V}$, then $F(z', u')$ is orthogonal to all $v \in \mathcal{V}$ by (21). Hence $F(z', u') = 0$ for all $(z', u') \in \Sigma$ and $F = 0$ in $\mathcal{H}_\pi$. Thus $\{F_{(z',u')}(\cdot)v : (z', u') \in \Sigma, v \in \mathcal{V}\}$ span a dense subspace of $\mathcal{H}_\pi$, and $\{f_{(z',u')}(\cdot)v : (z', u') \in \Sigma, v \in \mathcal{V}\}$ span also a dense subspace of $\mathfrak{SF}_\Gamma$.

THEOREM 2. *For $(z', u') \in \Sigma$ and $v \in \mathcal{V}$,*

$$\mathcal{K}_\pi f_{(z',u')}(\cdot)v = f_{(-z'^{-1}, -z'^{-1} \circ u')}(\cdot)\, \pi(k(u'))^*\, \pi(z')^* v \tag{22}$$

for all $f_{(z',u')}(\cdot)v \in \mathfrak{SF}_\Gamma$ given by (20).

PROOF. By (17),

$$((T_\pi(j))^* F)(z, u) = \pi(z)\, \pi(k(u))\, F(-z^{-1}, -z^{-1} \circ u).$$

By (21),

$$(T_\pi(j)) F_{(z',u')}(\cdot)v = F_{(-z'^{-1}, -z'^{-1} \circ u')}(\cdot)\, \pi(k(u'))^*\, \pi(z')^* v.$$

Since $F_{(z',u')} = \mathcal{T} f_{(z',u')}$ and $\mathcal{K}_\pi = \mathcal{T}^{-1} T_\pi(j) \mathcal{T}$, we have (22).

By Theorem 1, (22) gives an explicit form of the Hankel transform $\mathcal{K}_\pi$ on a dense subspace of $\mathfrak{SF}_\Gamma$.

We now adopt a lemma from [1].

LEMMA 1. (cf. [1], Lemma 5.5) *For $u \in V_1$ and invertible $z \in V_2$, we have*

$$\begin{aligned} &k(-u)\, r(z)^{-1}\, k(-iu)^*\, r(2Im\, z - B(u, \bar{u}))\, k(-iu)\, r(z^*)^{-1}\, k(-u)^* \\ &\quad = k(-iz^{-1} \circ u)^*\, r(2Im\, (-z^{-1}) - B(z^{-1} \circ u, \overline{z^{-1} \circ u}))\, k(-iz^{-1} \circ u). \end{aligned} \tag{23}$$

It follows from (23) that

$$\begin{aligned} &k(iu)^*\, r(z)\, k(u)\, k(-iz^{-1} \circ u)^*\, r(2Im\, (-z^{-1}) - B(z^{-1} \circ u, \overline{z^{-1} \circ u})) \\ &\qquad \times k(-iz^{-1} \circ u)\, k(u)^*\, r(z^*)\, k(iu) = r(2Im\, z - B(u, \bar{u})). \end{aligned} \tag{24}$$

We now show the square-integrability of S_π by the new description of $\mathcal{K}_\pi$.

Let dx and du be Lebesgue measures on J and V_1, and da Haar measure on A. $R(u_0)$ and $R(x_0)$ commute, and generate a subgroup N of G. Since $M = NA$ is a semidirect product,

$$\int_M f(m)\,dm = \int_A \int_{J\oplus V_1} f(R(x_0)\,R(u_0)\,a)\,dx_0\,du_0\,da \tag{25}$$

defines a right Haar measure on M, and this equality is valid for any nonnegative measurable function f. Moreover, the formula

$$\int_G f(g)\,dg = \int_M \int_{J\oplus V_1} f(m^{-1}\,j\,R(x_0)\,R(u_0))\,dm\,dx_0\,du_0 \tag{26}$$

is valid for all nonnegative measurable functions f.

LEMMA 2. (cf. [1], Theorem 7.8) *For any f_1 and f_2 in $\mathfrak{SF}_\Gamma$,*

$$\begin{aligned}\int_M |\langle S_\pi(m)f_1|f_2\rangle|^2\,dm &\le 2^{n_1}\,\pi^n\,||f_1||^2 \\ &\times \left(\int_{AK_{-\frac{1}{2}}} ||\Gamma_\pi^{\frac{1}{2}} f_2(a\,k(u_0))||^2\,e^{-(u_0|u_0)}\,\Delta(ae)^{-\mu-b}\,da\,du_0\right).\end{aligned} \tag{27}$$

THEOREM 3. *For any f_1 and f_2 in $\mathfrak{SF}_\Gamma$,*

$$\int_G |(S_\pi(g)f_1|f_2)|^2\,dg < \infty. \tag{28}$$

PROOF. By (26) and (27),

$$\begin{aligned}&\int_G |\langle S_\pi(g)\,f_1|f_2\rangle|^2\,dg \\ &\quad= \int_{J\oplus V_1}\int_M |\langle S_\pi(m)f_2|S_\pi(j\,R(u_0)\,R(x_0))\,f_1\rangle|^2\,dm\,dx_0\,du_0 \\ &\quad\le 2^{n_1}\,\pi^n\,||f_2||^2 \int_{J\oplus V_1}\left(\int_{AK_{-\frac{1}{2}}} ||\Gamma_\pi^{\frac{1}{2}}\,(S_\pi(j\,R(u_0)\,R(x_0))\,f_1)(a\,k(u))||^2\right. \\ &\qquad\left.\times\, e^{-(u|u)}\,\Delta(ae)^{-\mu-b}\,da\,du\right)dx_0\,du_0.\end{aligned} \tag{29}$$

We assume temporarily that $f_1 = f_{(z',u')}(\cdot)v$ for some $(z',u') \in \Sigma$, $v \in \mathcal{V}$. By (19) and (22), we have

$$\begin{aligned}&S_\pi(j\,R(u_0)\,R(x_0)f_{(z',u')})(a\,k(u))\,v \\ &\quad= \pi^{-\frac{n}{2}}\,e^{-i(ae|-(x_0+z'+\frac{i}{2}B(u_0,\bar{u}_0)+iB(u',\bar{u}_0))^{-1})} \\ &\qquad\times e^{(au|-(x_0+z'+\frac{i}{2}B(u_0,\bar{u}_0)+iB(u',\bar{u}_0))^{-1}\circ(u_0+u'))}\,\Gamma_\pi^{-1}\,\pi(a\,k(iu))^{-1} \\ &\qquad\times \pi(k(u_0+u'))^*\,\pi(x_0+z'+\frac{i}{2}B(u_0,\bar{u}_0)+iB(u',\bar{u}_0))^*\,\pi(k(iu_0))\,v.\end{aligned} \tag{30}$$

Denote

$$z_1 = x_0 + z' + \frac{i}{2}B(u_0,\bar{u}_0) + iB(u',\bar{u}_0) \tag{31}$$

and

$$u_1 = u_0 + u'. \tag{32}$$

Applying variable changes, (10), (11) and (13), we have

$$\begin{aligned}
&\int_{J\oplus V_1}\Big(\int_{AK_{-\frac{1}{2}}} ||\Gamma_\pi^{\frac{1}{2}}(S_\pi(j\,R(u_0)\,R(x_0))f_{(z',u')})(a\,k(u))\,v||^2 \\
&\qquad\qquad \times e^{-(u|u)}\,\Delta(ae)^{-\mu-b}\,da\,du\Big)\,dx_0\,du_0 \\
&\leq \pi^{-n}\,||\Gamma_\pi^{-\frac{1}{2}}||^2\,||\Gamma_\pi^{\frac{1}{2}}||^2\,||\pi(k(-iu'))||^2 \\
&\times \int_{J\oplus V_1}\Delta(-2Im\,z_1^{-1} - B(z_1^{-1}\circ u_1, \overline{z_1^{-1}\circ u_1}))^{\mu+b} \\
&\times ||\pi(r(-2Im\,z_1^{-1} - B(z_1^{-1}\circ u_1, \overline{z_1^{-1}\circ u_1}))^{\frac{1}{2}}k(-iz_1^{-1}\circ u_1)k(u_1)^* r(z_1^*)k(iu_1))v||^2 \\
&\times dx_0\,du_0.
\end{aligned}$$

By (31), (32) and a calculation, we have

$$2Im\,z_1 - B(u_1,\bar{u}_1) = 2Im\,z' - B(u',\bar{u}')$$

and

$$-2Im\,z_1^{-1} - B(z_1^{-1}\circ u_1, \overline{z_1^{-1}\circ u_1}) = z_1^{*-1}\,(2Im\,z' - B(u',\bar{u}'))\,z_1^{-1}.$$

It follows now from (24) that

$$\begin{aligned}
&\int_{J\oplus V_1}\Big(\int_{AK_{-\frac{1}{2}}} ||\Gamma_\pi^{\frac{1}{2}}(S_\pi(j\,R(u_0)\,R(x_0))f_{(z',u')})(a\,k(u))\,v||^2 \\
&\qquad\qquad e^{-(u|u)}\,\Delta(ae)^{-\mu-b}\,da\,du\Big)\,dx_0\,du_0 \\
&\leq \pi^{-n}\,||\Gamma_\pi^{-\frac{1}{2}}||^2\,||\Gamma_\pi^{\frac{1}{2}}||^2||\pi(k(-iu'))||^2\,||\pi(r(2Im\,z' - B(u',\bar{u}'))^{\frac{1}{2}})\,v||^2 \\
&\quad\times \Delta(2Im\,z' - B(u',\bar{u}'))^{\mu+b} \\
&\quad\times \int_{J\oplus V_1}\Big|\Delta(x_0 + z' + \frac{i}{2}B(u_0,\bar{u}_0) + iB(u',\bar{u}_0))\Big|^{-2\mu-2b} dx_0\,du_0.
\end{aligned} \tag{33}$$

By [5, (5.98)],

$$\int_{J\oplus V_1}\Big|\Delta(x_0 + z' + \frac{i}{2}B(u_0,\bar{u}_0) + iB(u',\bar{u}_0))\Big|^{-2\mu-2b} dx_0\,du_0 < \infty.$$

By (29), (33) and the fact that $(z',u') \in \Sigma$, (28) holds for $f_1 = f_{(z',u')}(\cdot)v$ and for all $f_2 \in \mathfrak{SF}_\Gamma$. Since the functions $f_{(z',u')}(\cdot)v$ for $(z',u') \in \Sigma$, $v \in \mathcal{V}$ span a dense subspace $(\mathfrak{SF}_\Gamma)^0$ of $\mathfrak{SF}_\Gamma$, it follows from Fatou's lemma that (28) also holds for all $f_1, f_2 \in \mathfrak{SF}_\Gamma$, which completes the proof of the theorem.

REFERENCES

1. H. Ding, *Weighted Laplace transforms and Bessel functions on Hermitian symmetric spaces*, preprint, St. Louis University, 1994.
2. H. Ding and K. I. Gross, *Operator-valued Bessel functions on Jordan algebras*, J. reine. angew. Math. **435** (1993), 157–196.
3. J. Dorfmeister, *Quasisymmetric Siegel domains and the automorphisms of homogeneous Siegel domains*, Amer. J. Math. **102** (1980), 537–563.
4. J. Faraut and A. Koranyi, *Function spaces and reproducing kernels on bounded symmetric domains*, J. Funct. Anal. **88** (1990), 64–89.
5. S. G. Gindikin, *Analysis on homogeneous domains*, Russian Math. Surveys **19** (1964), 1–90.
6. K. I. Gross and R. A. Kunze, *Bessel functions and representation theory, I*, J. Funct. Anal. **22** (1976), 73–105.
7. ———, *Bessel functions and representation theory, II: Holomorphic discrete series and metaplectic representations*, J. Funct. Anal. **25** (1977), 1–49.
8. S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.
9. M. Kashiwara and M. Vergne, *On the Segal-Shale-Weil representations and harmonic polynomials*, Invent. Math. **44** (1978), 1–47.
10. A. W. Knapp, *Representation Theory of Semisimple Lie Groups, an overview based on examples*, Princeton Univ. Press, Princeton, N.J., 1986.
11. O. Loos, *Bounded Symmetric Domains and Jordan Pairs*, Lecture notes, Univ. of California, Irvine, CA, 1977.
12. ———, *Jordan Pairs*, Lecture notes, Springer-Verlag, New York, 1975.
13. H. Rossi and M. Vergne, *Representations of certain solvable Lie groups on Hilbert spaces of holomorphic functions and the application to the holomorphic discrete series of a semisimple Lie group*, J. Funct. Anal. **13** (1973), 324–389.

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Fonctions Sphériques sur un Espace Symétrique Ordonné de Type Cayley

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en hommage à Ray Kunze, à l'occasion de son 65-ième anniversaire

ABSTRACT. For a special class of ordered symmetric spaces, called symmetric spaces of Cayley type, we give a formula for the spherical functions by using Jordan algebras methods, which is more explicit than in the general case, and we compute the function $c_{\mathcal{D}}(\lambda)$ occuring in the asymptotic behaviour of the spherical functions.

1. Introduction

Une *structure causale* sur une variété $\mathcal{M}$ est la donnée d'un champ de cônes C_x sur $\mathcal{M}$. Pour tout $x \in \mathcal{M}$, C_x est un cône régulier dans l'espace tangent $T_x(\mathcal{M})$, et C_x dépend différentiablement de x (un cône *régulier* est un cône convexe, fermé, pointu et d'intérieur non vide). Une courbe $\gamma : [\alpha, \beta] \to \mathcal{M}$ de classe C^1 est dite *causale* si, pour tout t, $\dot{\gamma}(t) \in C_{\gamma(t)}$. S'il n'existe pas de courbe causale fermée non triviale la structure causale est dite *globale*. Elle définit alors un ordre sur $\mathcal{M}$: on note $x \leq y$ s'il existe une courbe causale allant de x à y. Si les intervalles

$$D(x, y) = \{z \in \mathcal{M} \mid x \leq z \leq y\}$$

sont compacts, la variété causale $\mathcal{M}$ est dite *globalement hyperbolique*.

Supposons que $\mathcal{M}$ soit un espace homogène, $\mathcal{M} = G/H$, où G est un groupe de Lie connexe et H un sous-groupe fermé. La structure causale est dite invariante si

$$C_{\tau(g)x} = d\tau(g)_x(C_x),$$

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où $\tau(g)$ est la transformation $xH \mapsto gxH$. Soit $m_0 = eH$ le point de base. Une structure causale invariante sur $\mathcal{M} = G/H$ est déterminée par la donnée d'un cône régulier C dans $T_{m_0}(\mathcal{M})$ qui est invariant par H.

Supposons de plus que $\mathcal{M} = G/H$ soit un espace symétrique : il existe un automorphisme involutif σ de G tel que

$$(G^\sigma)_0 \subset H \subset G^\sigma,$$

où $G^\sigma = \{g \in G \mid \sigma(g) = g\}$ et $(G^\sigma)_0$ est la composante connexe de l'élément neutre dans G^σ. Notons $\mathfrak{g}$ et $\mathfrak{h}$ les algèbres de Lie de G et H,

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \sigma(X) = X\}$$

(nous notons également σ la différentielle de l'automorphisme involutif σ). Posons

$$\mathfrak{q} = \{X \in \mathfrak{g} \mid \sigma(X) = -X\} \simeq T_{m_0}(\mathcal{M}).$$

Ainsi une structure causale invariante sur $\mathcal{M} = G/H$ est déterminée par un cône régulier $C \subset \mathfrak{q}$ qui est invariant par $\mathrm{Ad}(H)$.

Nous supposons que G est semi-simple de centre fini, et que $(\mathfrak{g}, \mathfrak{h})$ est une paire symétrique irréductible. Il existe une involution de Cartan θ qui commutte avec σ, $\sigma \circ \theta = \theta \circ \sigma$. Soit

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

la décomposition de Cartan correspondante. Si $C \cap \mathfrak{k} = \{0\}$, la structure causale invariante correspondante est globale, et définit un ordre sur $\mathcal{M}$ pour lequel $\mathcal{M}$ est globalement hyperbolique. On dit que $\mathcal{M}$ est un espace symétrique ordonné. Posons

$$S = \{g \in G \mid g \cdot m_0 \geq m_0\}.$$

Alors S est un semi-groupe fermé et $S = \exp(C)H$ ([2], Theorem 4.1).

Il existe un sous-espace de Cartan $\mathfrak{a} \subset \mathfrak{p}$ qui est contenu dans $\mathfrak{p} \cap \mathfrak{q}$. Soit $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ le système de racines associé. On choisit dans Δ un système positif Δ^+ (voir [2], section 3) et on pose

$$\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}^\alpha, \quad \rho = \tfrac{1}{2} \sum_{\alpha \in \Delta^+} m_\alpha \alpha,$$
$$A = \exp \mathfrak{a}, \quad N = \exp \mathfrak{n}.$$

L'ensemble NAH est un ouvert de G. Pour $x \in NAH$, on pose

$$x = n \exp A(x) h.$$

On montre que $S \subset NAH$ ([2], Corollary 3.2). La fonction sphérique φ_λ ($\lambda \in \mathfrak{a}^*$) de l'espace symétrique $\mathcal{M} = G/H$ est définie sur l'intérieur S^0 de S par

$$\varphi_\lambda(x) = \int_H e^{\langle \rho - \lambda, A(hx) \rangle} \, dh.$$

Il existe un élément X_0 tel que les valeurs propres de $\mathrm{ad}(X_0)$ soient 1,0,-1. On note

$$\Delta_0^+ = \{\alpha \in \Delta^+ \mid \alpha(X_0) = 0\},$$
$$\Delta_1 = \{\alpha \in \Delta \mid \alpha(X_0) = 1\} \subset \Delta^+,$$
$$\mathfrak{n}_0 = \sum_{\alpha \in \Delta_0^+} \mathfrak{g}^\alpha,\ \mathfrak{n}_1 = \sum_{\alpha \in \Delta_1} \mathfrak{g}^\alpha,$$
$$N_0 = \exp \mathfrak{n}_0,\ N_1 = \exp \mathfrak{n}_1,\ \bar{N}_1 = \theta(N_1).$$

Ainsi $\Delta^+ = \Delta_0^+ \cup \Delta_1$, et $N = N_0 N_1$ est le produit semi-direct de N_0 et de N_1. L'ensemble $\mathcal{D} = \bar{N}_1 \cap NAH$ est un ouvert bormé de $\bar{N}_1$. L'intégrale qui définit φ_λ converge pour tout x de S^0 si et seulement si

$$c_{\mathcal{D}}(\lambda) = \int_{\mathcal{D}} e^{\langle \rho + \lambda, A(\bar{n}_1) \rangle} \quad d\bar{n}_1 < \infty$$

([2], Theorem 6.3, où cette intégrale est notée $c_\Omega(\lambda)$). Les fonctions sphériques φ_λ définissent des caractères de l'algèbre de Volterra $V(\mathcal{M})^\#$ et interviennent dans la transformation de Laplace sphérique (voir [2], section 8).

Dans cet article nous considérons une famille d'espaces symétriques ordonnés, appellés espaces symétriques ordonnés de type Cayley ([8], Definition 5.7). La géométrie et l'analyse de ces espaces peuvent être étudiées à l'aide des algèbres de Jordan (voir [6] et [7]). Nous explicitons la représentation intégrale des fonctions sphériques φ_λ et nous calculons la fonction $c_{\mathcal{D}}(\lambda)$.

2. L'espace symétrique ordonné $SL(2,\mathbb{R})/SO(1,1)$

L'espace $\mathcal{M} = SL(2,\mathbb{R})/SO(1,1)$ est l'exemple le plus simple d'un espace symétrique ordonné de type Cayley. Il peut être réalisé comme

$$\mathcal{M} = \{(x,y) \in P_1(\mathbb{R}) \times P_1(\mathbb{R}) \mid x \neq y\},$$

le groupe $G = SL(2,\mathbb{R})$ agissant sur $\mathcal{M}$ diagonalement. Le sous-groupe $H = SO(1,1)$ est le sous-groupe d'isotropie du point $m_0 = (1,-1)$. Si σ est l'automorphisme involutif de G défini par $\sigma(g) = JgJ$ avec

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

alors

$$H = G^\sigma = \{g \in G \mid \sigma(g) = g\}.$$

Ainsi $\mathcal{M}$ est un espace symétrique. Notons $\mathfrak{g}$ et $\mathfrak{h}$ les algèbres de Lie de G et H,

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \sigma(X) = X\} = \left\{ \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix} \;\middle|\; u \in \mathbb{R} \right\}.$$

Posons

$$\mathfrak{q} = \{X \in \mathfrak{g} \mid \sigma(X) = -X\},$$

alors

$$\mathfrak{q} = \left\{ \begin{pmatrix} v & u \\ -u & -v \end{pmatrix} \;\middle|\; u, v \in \mathbb{R} \right\}.$$

Soit $C \subset \mathfrak{q}$ le cône défini par

$$C = \left\{ \begin{pmatrix} v & u \\ -u & -v \end{pmatrix} \;\middle|\; u + v < 0,\ u - v > 0 \right\}.$$

Ce cône C est invariant par $\mathrm{Ad}H$. Il définit une structure causale invariante sur $\mathcal{M}$, et fait de $\mathcal{M}$ un espace symétrique ordonné globalement hyperbolique. Nous choisissons $\mathfrak{a} = \mathbb{R}X_0$, $\mathfrak{a}^+ = \mathbb{R}^+X_0$, avec

$$X_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

Alors $\Delta^+ = \{\alpha\}$, avec $\alpha(tX_0) = t$, et

$$A = \left\{ a_t = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\},$$

$$N = \left\{ n_u = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \;\middle|\; u \in \mathbb{R} \right\}.$$

L'orbite

$$NA \cdot m_0 = \{(e^t + u, -e^t + u) \mid t, u \in \mathbb{R}\}$$

s'identifie au demi-plan $\{(u, v) \mid v > 0\}$ $(v = e^t)$. Considérons sur $W = \mathbb{R}^2$ la structure d'algèbre définie par

$$(u_1, v_1)(u_2, v_2) = (u_1u_2 + v_1v_2, u_1v_2 + u_2v_1).$$

Identifions à $\mathbb{R}$ la sous-algèbre $\{(u, 0) \mid u \in \mathbb{R}\}$ de W, et posons $j = (0, 1)$. Alors $j^2 = 1$ et $(u, v) = u + jv$. Cette algèbre est isomorphe à l'algèbre produit $\mathbb{R} \times \mathbb{R}$, l'isomorphisme étant défini par

$$w = u + jv \mapsto (u + v, u - v).$$

Un élément $w = u + jv$ de W est inversible si $u^2 - v^2 \neq 0$. Si

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G = SL(2, \mathbb{R})$$

on pose

$$g \cdot w = (aw + b)(cw + d)^{-1},$$

l'inverse étant pris au sens de l'algèbre W ($g \cdot w$ n'est défini que si $cw + d$ est inversible). Cette action de $G = SL(2, \mathbb{R})$ correspond à l'action de G sur l'espace symétrique $\mathcal{M}$.

Par analogie avec le corps des nombres complexes nous noterons, si $w = u + jv$,

$$\Re w = u,\ \Im w = v.$$

Si $w = u + jv = n_u a_t \cdot m_0$, alors $t = \log(\Im w)$, et $A(w) = \log(\Im w) \cdot X_0$. Pour $\Re s > 0$ soit Ξ_s la fonction définie sur

$$S^0 \cdot m_0 = \{(u, v) \mid 0 < v < 1,\ |u| < 1 - v\}$$

par

$$\Xi_s(w) = \int_H \big(\Im(h \cdot w)\big)^s dh.$$

Alors $\varphi_{\lambda-\rho}(w) = \Xi_s(w)$, si $\lambda(X_0) = -\lambda_1$, $s = \lambda_1 + \frac{1}{2}$. Un élément de H s'écrit

$$h = \pm \begin{pmatrix} \operatorname{ch}\frac{\theta}{2} & \operatorname{sh}\frac{\theta}{2} \\ \operatorname{sh}\frac{\theta}{2} & \operatorname{ch}\frac{\theta}{2} \end{pmatrix},$$

et un calcule simple donne

$$\Im(h \cdot jv) = \Big(\operatorname{ch}^2\frac{\theta}{2} v^{-1} - \operatorname{sh}^2\frac{\theta}{2} v\Big)^{-1}.$$

Par suite, pour $0 < v < 1$,

$$\Xi_s(jv) = \int_{-\infty}^{\infty} \Big(\operatorname{ch}^2\frac{\theta}{2} v^{-1} - \operatorname{sh}^2\frac{\theta}{2} v\Big)^{-s} d\theta.$$

Cette intégrale converge pour $\Re s > 0$. Elle peut s'écrire

$$\Xi_s(je^{-t}) = \int_{-\infty}^{\infty} (\operatorname{ch} t + \operatorname{ch}\theta \operatorname{sh} t)^{-s} d\theta.$$

La fonction Ξ_s s'exprime à l'aide d'une fonction de Legendre de seconde espèce : pour $t > 0$,

$$\Xi_s(je^{-t}) = 2Q_{s-1}(\operatorname{ch} t).$$

et

$$\varphi_\lambda(je^{-t}) = 2Q_{\lambda_1 - \frac{1}{2}}(\operatorname{ch} t).$$

Nous avons

$$\bar{N}_1 = \bar{N} = \Big\{\bar{n}_u = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \;\Big|\; u \in \mathbb{R}\Big\},$$

et $\Im(\bar{n}_u \cdot j) = (1 - u^2)^{-1} > 0$ lorsque $-1 < u < 1$. On en déduit que

$$\mathcal{D} = \bar{N} \cap NAH = \{\bar{n}_u \mid -1 < u < 1\},$$

et

$$c_{\mathcal{D}}(\lambda) = \int_{-1}^{1} (1 - u^2)^{\lambda_1 - \frac{1}{2}}\, du = B(\lambda_1 + \tfrac{1}{2}, \tfrac{1}{2}),$$

où B désigne la fonction beta d'Euler.

3. Cônes symétriques et algèbres de Jordan

Soit Ω un cône ouvert convexe dans un espace vectoriel euclidien V de dimension n. On suppose que $\bar{\Omega}$ est pointu. Soit $G(\Omega)$ le groupe des transformations linéaires qui préservent Ω,

$$G(\Omega) = \{g \in GL(V) \mid g\Omega = \Omega\}.$$

Le cône Ω est dit *homogène* si $G(\Omega)$ opère transitivement sur Ω. Le cône ouvert dual Ω^* est défini par

$$\Omega^* = \{x \in V \mid \forall y \in \bar{\Omega} \setminus \{0\},\ (x|y) > 0\}.$$

Le cône Ω est dit *autodual* si $\Omega^* = \Omega$. S'il est homogène et autodual, le cône Ω est dit *symétrique*.

Une *algèbre de Jordan euclidienne* est un espace vectoriel euclidien V muni d'un produit vérifiant

$$(1)\quad xy = yx,$$
$$(2)\quad x(x^2y) = x^2(xy),$$
$$(3)\quad (xy|z) = (y|xz).$$

Koecher [5] et Vinberg [9] ont montré que l'intérieur du cône des carrés d'une algèbre de Jordan euclidienne est un cône symétrique, et que tout cône symétrique peut être obtenu de cette façon.

Par exemple si $V = Sym(m, \mathbb{R})$ est l'espace des matrices symétriques réelles $m \times m$ muni du produit de Jordan

$$x \circ y = \tfrac{1}{2}(xy + yx),$$

alors V est une algèbre de Jordan euclidienne relativement au produit scalaire défini par

$$(x|y) = \mathrm{tr}(xy),$$

et le cône symétrique Ω est le cône des matrices symétriques définies positives.

Pour $x \in V$ notons $L(x)$ et $P(x)$ les endomorphismes autoadjoints de V définis par

$$L(x)y = xy,$$
$$P(x) = 2L(x)^2 - L(x^2).$$

Si $V = Sym(m, \mathbb{R})$, $P(x)y = xyx$. Pour x dans Ω, $P(x)$ est défini positif. Considérons sur Ω la structure riemanienne invariante définie par

$$G_x(u, v) = \big(P(x^{-1})u|v\big),\ x \in \Omega,\ u, v \in V.$$

Cette structure riemannienne est invariante par $G(\Omega)$, et fait de Ω un espace riemannien symétrique qui s'identifie à l'espace quotient G_0/K_0 où G_0 est la composante connexe neutre de $G(\Omega)$, et

$$K_0 = \{g \in G_0 \mid ge = e\},$$

e désignant l'élément neutre de V.

Nous supposons que l'algèbre de Jordan V est simple, c'est à dire qu'elle n'a pas d'idéal non trivial. Soit r le rang de V et soit $\{c_1, \dots, c_r\}$ un système complet d'idempotents primitifs deux à deux orthogonaux. Tout élément x de V s'écrit

$$x = k \sum_{j=1}^{r} \lambda_j c_j, \ k \in K_0, \ \lambda_j \in \mathbb{R}.$$

Le déterminant Δ est défini par $\Delta(x) = \prod \lambda_j$, et la trace par $\mathrm{tr}(x) = \sum \lambda_j$. Δ est un polynôme homogène de degré r, et on peut supposer que le produit scalaire est défini par $(x|y) = \mathrm{tr}(xy)$. Si c est un idempotent, l'espace $V(c,1) = \{x \in V \mid cx = x\}$ est une sous-algèbre de V, et pour $1 \leq j \leq r$, $V_j = V(c_1 + \cdots + c_j, 1)$ est une sous-algèbre de V de rang j. Le déterminant mineur principal Δ_j est défini par

$$\Delta_j(x) = \Delta_{V_j}(x^{(j)}),$$

où Δ_{V_j} désigne le déteminant relatif à la sous-algèbre V_j, et $x^{(j)}$ est la projection orthogonale de x sur V_j. Les mineurs principaux sont positifs sur Ω. Pour $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{C}^r$, et pour $x \in \Omega$, on pose

$$\Delta_{\mathbf{s}}(x) = \Delta_1(x)^{s_1 - s_2} \Delta_2(x)^{s_2 - s_3} \dots \Delta_r(x)^{s_r}.$$

Les fonctions sphériques $\Phi_{\mathbf{s}}$ du cône symétrique $\Omega = G_0/K_0$ sont données par la représentation intégrale suivante

$$\Phi_{\mathbf{s}}(x) = \int_{K_0} \Delta_{\mathbf{s}}(kx) dk.$$

Au sujet de la géométrie et de l'analyse des cônes symétriques on peut consulter [3].

4. Espaces symétriques ordonnés de type Cayley

Comme dans la section 3, soit Ω le cône symétrique associé à une algèbre de Jordan simple euclidienne V, et soit T_Ω le tube dans l'espace complexifié $V^{\mathbb{C}}$ de base Ω,

$$T_\Omega = V + i\Omega = \{z = x + iy \mid x \in V, \ y \in \Omega\}.$$

Le groupe G des automorphismes holomorphes de T_Ω est engendré par le groupe affine P des transformations

$$z \mapsto gz + a \quad (g \in G(\Omega), \ a \in V),$$

et par la 'symétrie' s,

$$s(z) = -z^{-1}.$$

L'espace homogène $M = G/P$ est compact, et l'application

$$V \to M,$$
$$a \mapsto gP,$$

avec

$$g(z) = s(z) + a,$$

définit un plongement de V dans M d'image dense. Ainsi M est une compactification de V (parfois appelée compactification conforme). L'espace M est isomorphe à la frontière de Chilov de la réalisation bornée D de T_Ω. Le groupe G opère sur M, et donc sur $M \times M$ diagonalement. Soit $\mathcal{M}$ l'orbite de $m_0 = (e, -e)$ dans $M \times M$. On peut montrer que

$$\mathcal{M} \cap (V \times V) = \{(x, y) \in V \times V \mid \Delta(x - y) \neq 0\}$$

(voir [6], lemme IV.4.1, et [7]). Soit H le stabilisateur de $m_0 = (e, -e)$ dans G. La paire (G, H) est une paire symétrique relativement à l'automorphisme involutif σ de G défini par

$$\sigma(g) = \nu \circ g \circ \nu,$$

où ν est la conjugaison de T_Ω définie par

$$\nu(z) = \bar{z}^{-1}.$$

Par exemple, si $V = Sym(m, \mathbb{R})$, alors $\mathcal{M} \simeq Sp(m, \mathbb{R})/GL(m, \mathbb{R})$.

L'algèbre de Lie $\mathfrak{g}$ de G est l'algèbre des champs de vecteurs X sur V de la forme

$$X(z) = u + Tz - P(z)v,$$

avec $u, v \in V$, $T \in \mathfrak{g}_0$, l'algèbre de Lie de G_0. Ainsi un élément X de $\mathfrak{g}$ est un triplet $(u, T, v) \in V \times \mathfrak{g}_0 \times V$. L'involution σ opère dans $\mathfrak{g}$ par

$$\sigma(u, T, v) = (v, -T^*, u),$$

et, si $\mathfrak{h}$ désigne l'algèbre de Lie de H,

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \sigma(X) = X\} = \{(u, T, u) \mid u \in V, T \in \mathfrak{k}_0\},$$

où $\mathfrak{k}_0$ désigne l'algèbre de Lie de K_0. Posons

$$\mathfrak{q} = \{X \in \mathfrak{g} \mid \sigma(X) = -X\} = \{(u, 2L(v), -u) \mid u, v \in V\}.$$

Sous l'action de $\mathrm{Ad}(H)$, l'espace $\mathfrak{q}$ se décompose en deux sous-espaces irréductibles,

$$\mathfrak{q}_+ = \{(u, 2L(u), -u) \mid u \in V\} \simeq T_e(M) \simeq V,$$
$$\mathfrak{q}_- = \{(u, -2L(u), -u) \mid u \in V\} \simeq T_{-e}(M) \simeq V.$$

Le cône $C \subset \mathfrak{q}$ défini par

$$C = \{(u, 2L(v), -u) \mid u + v \in -\bar{\Omega},\ u - v \in \bar{\Omega}\}$$

est invariant par $\mathrm{Ad}(H)$. Il définit sur $\mathcal{M}$ une structure causale invariante et en fait un espace symétrique ordonné globalement hyperbolique (voir [2], Theorem 4.1).

Soit $\mathfrak{a}$ le sous-espace de Cartan suivant

$$\mathfrak{a} = \{(0, L(t), 0) \mid t = t_1 c_1 + \cdots + t_r c_r\} \simeq \mathbb{R}^r.$$

Nous choisissons la chambre de Weyl positive

$$\mathfrak{a}^+ = \{(0, L(t), 0) \mid 0 < t_1 < t_2 < \cdots < t_r\},$$

et prenons $X_0 = (0, I, 0)$. Soit γ_i l'élément de $\mathfrak{a}^*$ défini par

$$\gamma_i\big((0, L(t), 0)\big) = t_i \quad (1 \leq i \leq r).$$

Alors

$$\Delta_0^+ = \{\tfrac{1}{2}(\gamma_j - \gamma_i) \mid i < j\},$$
$$\Delta_1 = \{\gamma_i \mid 1 \leq i \leq r\} \cup \{\tfrac{1}{2}(\gamma_i + \gamma_j) \mid i < j\}.$$

La multiplicité de la racine γ_i est égale à 1, et on note d la multiplicité commune des racines $\frac{1}{2}(\gamma_j - \gamma_i)$ et $\frac{1}{2}(\gamma_i + \gamma_j)$, par suite

$$\rho = \sum_{i=1}^{r} \big((2i - r - 1)\frac{d}{4} + \frac{n}{2r}\big)\gamma_i \quad (n = \dim V).$$

(Noter que $n = r + \frac{d}{2}r(r-1)$.) Les sous-groupes A et N_0 sont contenus dans G_0,

$$A = \{P(\exp \frac{t}{2}) \mid t = t_1 c_1 + \cdots + c_r t_r\},$$

N_0 est le groupe triangulaire strict (noté N dans [3]), et AN_0 est le groupe triangulaire (noté T dans [3]). Le sous-groupe N_1 est le sous-groupe des translations

$$z \mapsto z + u \quad (u \in V).$$

Soit τ l'élément de G défini par

$$\tau(z) = (e + z)(e - z)^{-1} = -e - 2(z - e)^{-1},$$

d'inverse

$$\tau^{-1}(z) = (z - e)(z + e)^{-1} = e - 2(z + e)^{-1}.$$

On montre que $H = \tau \circ G(\Omega) \circ \tau^{-1}$ (voir [6], théorème IV.4.3, et [7]).

LEMME 4.1. *Soit $\omega \in \Omega$ tel que $\Delta(\omega^2 - e) \neq 0$, et soit $h = \tau \circ P(\omega) \circ \tau^{-1} \in H$, alors*

$$h(x) = -a + P(b)(a - x)^{-1},$$

avec

$$a = (\omega + \omega^{-1})(\omega - \omega^{-1})^{-1}, \ b = 2(\omega - \omega^{-1})^{-1}.$$

DÉMONSTRATION. Posons $y = \tau^{-1}(x)$, alors (voir [3], Proposition II.3.3 (ii))

$$\begin{aligned} h(x) &= \tau\big(P(\omega)y\big) \\ &= -e + 2\big(e - P(\omega)y\big)^{-1} \\ &= -e + 2P(\omega^{-1})(\omega^{-2} - y)^{-1} \\ &= -e + P(\omega^{-1})\big((x + e)^{-1} - (a + e)^{-1}\big)^{-1}. \end{aligned}$$

En utilisant l'identité de Hua ([3], Chapter II, Ex.5),

$$u^{-1} - v^{-1} = \big(u + P(u)(v - u)^{-1}\big)^{-1},$$

nous obtenons

$$h(x) = -e - P(\omega^{-1})\big((a + e) + P(a + e)(x - a)^{-1}\big).$$

D'autre part

$$e + P(\omega^{-1})(a+e) = a,\ P(\omega^{-1})P(a+e) = P(b). \qquad \square$$

Considérons dans $\mathcal{M}$ l'ouvert

$$NA \cdot m_0 = \{(ge+u, -ge+u) \mid g \in N_0 A,\ u \in V\}.$$

En posant $v = ge$ cet ouvert s'identifie au tube $V \times \Omega \subset V \times V$. Nous munissons $W = V \times V$ de la structure d'algèbre de Jordan définie par

$$(u_1, v_1)(u_2, v_2) = (u_1u_2 + v_1v_2, u_1v_2 + v_1u_2).$$

Nous identifions la sous-algèbre $\{(u,0) \mid u \in V\}$ à l'algèbre V et posons $j = (0,e)$. Alors $j^2 = e$, et nous pouvons écrire $(u,v) = u + jv$. Cette algèbre W est isomorphe à l'algèbre produit $V \times V$, l'isomorphisme étant donné par

$$u + jv \mapsto (u+v, u-v) = (x,y) \in V \times V.$$

Nous noterons

$$\Re(u+jv) = u,\ \Im(u+jv) = v.$$

LEMME 4.2. *Si v est inversible dans V, et $u+jv$ inversible dans W, alors*

$$\Im(u+jv)^{-1} = \left(v - P(u)v^{-1}\right)^{-1}.$$

DÉMONSTRATION. Supposons d'abord que $u = e$, alors

$$(e+jv)^{-1} = (e-jv)(e-v^2)^{-1},$$

et

$$\begin{aligned}\Im(e+jv)^{-1} &= -v(e-v^2)^{-1} \\ &= (v - v^{-1})^{-1}.\end{aligned}$$

Supposons que $u \in \Omega$, alors

$$u + jv = P(u^{\frac{1}{2}})\left(e + jP(u^{-\frac{1}{2}})v\right),$$

et

$$\begin{aligned}\Im(u+jv)^{-1} &= P(u^{-\frac{1}{2}})\left(P(u^{-\frac{1}{2}})v - P(u^{\frac{1}{2}})v^{-1}\right)^{-1} \\ &= \left(v - P(u)v^{-1}\right)^{-1}.\end{aligned}$$

Les deux membres de l'égalité à démontrer sont des fonctions rationnelles qui sont égales pour $u \in \Omega$, elles sont donc égales pour tout u. $\square$

LEMME 4.3. *Soit $h = \tau \circ P(\omega) \circ \tau^{-1}$, avec $\omega \in \Omega$, $\Delta(e - \omega^2) \neq 0$, alors*

$$\Im h(jv) = \left(P(\alpha)v^{-1} - P(\beta)v\right)^{-1},$$

avec

$$\alpha = \tfrac{1}{2}(\omega + \omega^{-1}),\ \beta = \tfrac{1}{2}(\omega - \omega^{-1}).$$

DÉMONSTRATION. D'après le lemme 4.1,

$$h(jv) = -a + P(b)(a - jv)^{-1},$$

et d'après le lemme 4.2,

$$\Im h(jv) = P(b^{-1})\big(-v + P(a)v^{-1}\big)^{-1}.$$

De plus

$$P(b^{-1})P(a) = P(b^{-1}a) = P\big(\tfrac{1}{2}(\omega + \omega^{-1})\big). \qquad \square$$

Le lemme précédent permet d'expliciter la représentation intégrale de la fonction sphérique φ_λ ([2], section 5).

THÉORÈME 4.4. *Posons*

$$\rho - \lambda = \sum_{i=1}^{r} s_i\gamma_i,$$

alors

$$\varphi_\lambda(jv) = \Xi_{\mathbf{s}}(jv) = \int_\Omega \Phi_{\mathbf{s}}\Big(\big(P(\alpha)v - P(\beta)v^{-1}\big)^{-1}\Big)\Delta(\omega)^{-\frac{n}{r}}d\omega,$$

avec

$$\alpha = \tfrac{1}{2}(\omega^{\frac{1}{2}} + \omega^{-\frac{1}{2}}),\ \beta = \tfrac{1}{2}(\omega^{\frac{1}{2}} - \omega^{-\frac{1}{2}}).$$

DÉMONSTRATION. Puisque

$$e^{\langle\rho-\lambda, A(w)\rangle} = \Delta_{\mathbf{s}}(\Im w),$$

la représentation intégrale de φ_λ s'écrit

$$\varphi_\lambda(w) = \int_H \Delta_{\mathbf{s}}\big(\Im(h\cdot w)\big)dh = \int_{K_0\backslash H} \Phi_{\mathbf{s}}\big(\Im(h\cdot w)\big)d\dot{h}.$$

L'espace $K_0 \setminus H$ s'identifie au cône Ω par l'application

$$h = k_0\,\tau \circ P(\omega^{\frac{1}{2}}) \circ \tau^{-1} \mapsto \omega,$$

et le résultat annoncé se déduit du lemme 4.3. $\square$

Cette représentation intégrale peut servir à définir les fonctions de Legendre généralisées de deuxième espéce $Q_{\mathbf{s}}$ sur une algèbre de Jordan

$$Q_{\mathbf{s}}\big(\tfrac{1}{2}(v + v^{-1})\big) = \Xi_{\mathbf{s}}(jv^{-1}),$$

pour $v \in e + \Omega$.

5. La fonction $c_{\mathcal{D}}(\lambda)$

D'après le théorème 6.3 de [2] l'intégrale qui définit φ_λ (λ réel) converge si et seulement si

$$c_{\mathcal{D}}(\lambda) = \int_{\mathcal{D}} a_H(\bar{n}_1)^{\rho+\lambda}d\bar{n}_1 < \infty,$$

(notée $c_\Omega(\lambda)$ dans [2]) où $\mathcal{D}$ désigne la boule unité de V pour la norme spectrale

$$\mathcal{D} = \{x \in V \mid |x| < 1\}.$$

Un élément $\bar{n}_1$ de N_1 s'écrit

$$\bar{n}_1 \cdot z = (z^{-1} - u)^{-1} \quad (u \in V).$$

Appliquons $\bar{n}_1$ à je. D'après le lemme 4.2

$$\Im(\bar{n}_1 \cdot je) = \Im(je - u)^{-1} = (e - u^2)^{-1}.$$

On en déduit que $c_{\mathcal{D}}(\lambda)$ peut s'écrire

$$c_{\mathcal{D}}(\lambda) = \int_{|u|<1} \Delta_{\mathbf{s}}(e - u^2)du,$$

avec

$$\lambda + \rho = -\sum_{i=1}^{r} s_{r-i+1}\gamma_i.$$

THÉORÈME 5.1. *Posons*

$$\lambda = -\sum_{i=1}^{r} \lambda_i \gamma_i,$$

alors

$$c_{\mathcal{D}}(\lambda) = c \prod_{j=1}^{r} B(\lambda_j + \frac{1}{2}, \frac{1}{2}) \prod_{j<k} B(\lambda_j + \lambda_k - \frac{d}{2} + 1, \frac{d}{2}).$$

COROLLAIRE 5.2. *L'intégrale qui définit φ_λ est convergente si et seulement si*

$$\Re\lambda_j > -\frac{1}{2}, \ \Re(\lambda_j + \lambda_k) > \frac{d}{2} - 1.$$

DÉMONSTRATION. Posons

$$J_r(\mathbf{s}) = \int_{|x|<1} \Delta_{\mathbf{s}}(e - x^2)dx.$$

Nous allons calculer $J_r(\mathbf{s})$ par récurrence sur r, en adaptant une méthode due à Hua ([4], Theorem 2.1.1). Considérons la décomposition de Peirce de V relativement à l'idempotent c_r,

$$\begin{aligned} V =& V(c_r, 1) + V(c_r, \tfrac{1}{2}) + V(c_r, 0), \\ x =& tc_r + \xi + x_0 \quad (t \in \mathbb{R}). \end{aligned}$$

Soit ϕ la représentation de $V_0 = V(c_r, 0)$ dans $E = V(c_r, \frac{1}{2})$ définie par

$$\phi(x_0)\xi = 2x_0\xi.$$

La forme quadratique associée à ϕ, définie sur E à valeurs dans V_0, définie par

$$(Q(\xi)|x_0) = (\phi(x_0)\xi|\xi),$$

vaut

$$Q(\xi) = 2(\xi^2)_0.$$

Notons $e_k = c_1 + \cdots + c_k$ $(1 \leq k \leq r)$. La décomposition de Peirce de $e - x^2$ s'écrit

$$\begin{aligned} e - x^2 =& (1 - t^2 - \tfrac{1}{2}\|\xi\|^2)c_r \\ & - \big(\phi(x_0)\xi + t\xi\big) \\ & + \big(e_{r-1} - x_0^2 - \tfrac{1}{2}Q(\xi)\big). \end{aligned}$$

En utilisant la relation

$$\Delta(x) = \Delta_{r-1}(x_0)\Big(t - \tfrac{1}{2}\big(\phi(x_0^{-1})\xi|\xi\big)\Big)$$

nous obtenons

$$\Delta(e - x^2) = \Delta_{r-1}\big(e_{r-1} - x_0^2 - \tfrac{1}{2}Q(\xi)\big)p(t),$$

où $p(t)$ est un polynôme du second degre, $p(t) = at^2 + 2bt + c$, avec

$$a = -1 - \Big(\big(e_{r-1} - x_0^2 - \tfrac{1}{2}Q(\xi)\big)^{-1}\xi|\xi\Big).$$

Soit τ_0 l'élément du groupe triangulaire T_0 de l'algèbre de Jordan V_0 associé au système d'idempotents $c_1, \ldots, c_{r-1}$ tel que

$$e_{r-1} - x_0^2 = \tau_0 e_{r-1}.$$

Il existe une transformation linéaire $\tilde{\tau}_0$ de l'espace E vérifiant

$$Q(\tilde{\tau}_0\xi) = \tau_0 Q(\xi), \ \phi(\tau_0 x_0) = \tilde{\tau}_0\phi(x_0)\tilde{\tau}_0^*.$$

Posons $\xi = \sqrt{2}\tilde{\tau}_0\eta$,

$$e_{r-1} - x_0^2 - \tfrac{1}{2}Q(\xi) = \tau_0\big(e_{r-1} - Q(\xi)\big).$$

Tenant compte du fait que $Q(\eta)$ est un élément de rang ≤ 1 nous obtenons

$$\begin{aligned} a &= -1 - \Big(\phi\big(e_{r-1} - Q(\eta)\big)^{-1}\eta|\eta\Big) \\ &= -1 - \frac{\|\eta\|^2}{1 - \|\eta\|^2} = \frac{1}{1 - \|\eta\|^2} \\ &= -\Delta_{r-1}\big(e_{r-1} - Q(\eta)\big)^{-1}. \end{aligned}$$

On peut montrer que $b^2 - ac = 1$. Notons que

$$d\xi = 2^{\frac{d}{2}}\Delta_{r-1}(e_{r-1} - x_0^2)^{\frac{d}{2}}d\eta.$$

LEMME 5.3. *Soit E un espace euclidien de dimension N. Pour $\Re s > -1$,*

$$\int_{\|x\|<1}(1 - \|x\|^2)^s dx = \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}B(s+1, \frac{N}{2}).$$

LEMME 5.4. *Soit $p(t) = at^2 + 2bt + c$ un polynôme du second degré tel que $a < 0$, $b^2 - ac = 1$. Soient α et β les racines de p. Pour $\Re s > -1$,*

$$\int_\alpha^\beta p(t)^s dt = B(s+1, \tfrac{1}{2})|a|^{-s-1}.$$

Nous intégrons d'abord par rapport à t en utilisant le lemme 5.4. Nous obtenons

$$J_r(\mathbf{s}) = J_{r-1}(\mathbf{s}^0 + \frac{d}{2})B(s_r+1, \tfrac{1}{2})2^{\frac{d}{2}} \int_{\|\eta\|<1} \Delta_\sigma^{(r-1)}\big(e_{r-1} - Q(\eta)\big)d\eta,$$

avec $\mathbf{s}^0 = (s_1, \dots, s_{r-1})$, $\sigma_j = s_j + s_r + 1$ $(1 \le j \le r-1)$, et

$$\Delta_\sigma^{(k)} = \Delta_1^{\sigma_1 - \sigma_2} \dots \Delta_k^{\sigma_k} \ (1 \le k \le r).$$

Pour le calcul de l'intégrale en η nous posons

$$\eta_1 = \phi(c_{r-1})\eta, \ \eta_0 = \eta - \eta_1.$$

Alors la composante de $Q(\eta)$ dans $V_0(c_{r-1}, 0)$ est égale à $Q(\eta_0)$. Ainsi

$$\Delta_\sigma^{(r-1)}\big(e_{r-1} - Q(\eta)\big) = \Delta_{\sigma-\sigma_{r-1}}^{(r-2)}\big(e_{r-2} - Q(\eta_0)\big)\Delta_{r-1}\big(e_{r-1} - Q(\eta)\big)^{\sigma_{r-1}}.$$

Puisque $Q(\eta)$ est un élément de rang ≤ 1,

$$\Delta_{r-1}\big(e_{r-1} - Q(\eta)\big) = 1 - \|\eta\|^2.$$

Posons

$$\eta_1 = (1 - \|\eta_0\|^2)^{\frac{1}{2}} v, \ v \in \phi(c_{r-1})E,$$

alors

$$\int_{\|\eta\|<1} (1 - \|\eta_0\|^2 - \|\eta_1\|^2)^{\sigma_{r-1}} d\eta_1$$
$$= (1 - \|\eta_0\|^2)^{\sigma_{r-1} + \frac{d}{2}} \int_{\|v\|<1} (1 - \|v\|^2)^{\sigma_{r-1}} dv,$$

et, d'après le lemme 5.3,

$$= c(1 - \|\eta_0\|^2)^{\sigma_{r-1} + \frac{d}{2}} B(\sigma_{r-1} + 1, \frac{d}{2}).$$

On montre ainsi par récurrence que

$$\int_{\|\eta\|<1} \Delta_\sigma^{(r-1)}\big(e_{r-1} - Q(\eta)\big)d\eta = c\prod_{j=1}^{r-1} B\big(\sigma_j + (r-j-1)\frac{d}{2} + 1, \frac{d}{2}\big).$$

Finalement, puisque

$$s_j = \lambda_{r-j-1} - \frac{d}{2}(r + 1 - 2j) - \frac{n}{2r},$$

nous obtenons

$$J_r(\mathbf{s}) = c\prod_{i=1}^{r} B(\lambda_i + \frac{1}{2}, \frac{1}{2})\prod_{i<j} B(\lambda_i + \lambda_j - \frac{d}{2} + 1, \frac{d}{2}). \qquad \square$$

Nous concluons par deux remarques. Dans [2] nous avons déterminé la fonction $c_{\mathcal{D}}$ dans le cas des espaces symétriques ordonnés de rang un, et dans le cas des espaces ordonnés de type $G_{\mathbb{C}}/G_{\mathbb{R}}$, dits de Olshanski. Dans tous les cas connus

$$c_{\mathcal{D}}(\lambda) = c \prod_{\alpha \in -\Delta_1} B\Big(\frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} - \frac{m_\alpha}{2} + 1, \frac{m_\alpha}{2}\Big).$$

Nous conjecturons que cette formule est vraie dans le cas général.

Le dual riemannien de l'espace symétrique ordonné $\mathcal{M}$ est le tube $T_\Omega = G/K$, où $K = \{g \in G \mid g \cdot ie = ie\}$. Dans [1] nous avons étudié les fonctions sphériques de T_Ω par des méthodes analogues à celle du présent article. Il serait interessant de mieux comprendre la relation qui existe entre l'analyse harmonique de l'espace symétrique ordonné $\mathcal{M}$ et celle de son dual riemannien T_Ω.

References

1. J. Faraut, *Fonctions de Legendre sur une algèbre de Jordan*, CWI Quarterly **5** (1992), 309–320.
2. J. Faraut, J. Hilgert and G. Ólafsson, *Spherical functions on ordered symmetric spaces*, Ann. Inst. Fourier **44** (1994), 927–966.
3. J. Faraut and A. Korányi, *Analysis on Symmetric Cones*, Oxford Mathematical Monograph, Oxford, 1994.
4. L. K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains*, Amer. Math. Soc., Providence, R.I., 1963.
5. M. Koecher, *Die Geodätischen von Positivitätsbereichen*, Math. Ann. **135** (1958), 192–202.
6. K. Koufany, *Semi-groupe de Lie associé à une algèbre de Jordan euclidienne*, Thèse, Université de Nancy, 1993.
7. ———, *Réalisation des espaces symétriques de type Cayley*, C. R. Acad. Sci. Paris, Série I **318** (1994), 425–428.
8. G. Ólafsson, *Symmetric spaces of Hermitian type*, Differential Geometry and its Applications **1** (1991), 195–233.
9. E. B. Vinberg, *Homogeneous cones*, Soviet Math. Dokl. **1** (1960), 787–790.

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On Derived Intertwining Norms for the Lorentz Group

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ABSTRACT. We construct explicit unitary structures on the irreducible subspaces of reducible spherical principal series of $SO_e(1,n)$ using the binomial theorem, the theory of spherical harmonics, and some results from Fourier analysis on Euclidean space.

1. Introduction

We start with an elementary proof of the unitarity of complementary series for $SO_e(1,n)$ based on a technique due to Hansen [9]. We then extend this method to produce unitary structures on the irreducible subspaces of a reducible spherical principal series representation (Theorem 1). This is done by differentiating the intertwining operators, hence the name derived intertwining norm. Using a generalisation of Pitt's inequality for Fourier transforms of functions with some zero moments, we identify these subspaces with some Sobolev spaces on $\mathbf{R}^{n-1}$ (Theorem 2.) The final part of the paper is concerned with using Cauchy-Szegö operators to give alternative realisations of these representations, including the unitary structure (Theorem 3.) The content of this paper gives a new proof of Theorem 9 in the paper by Davis, Gilbert, and Kunze [2].

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2. Spherical Principal Series for the Lorentz group

2.1. Notation. We set up some standard material, for more details see for example [7,10]. Vectors in $\mathbf{R}^{n+1}$ are written as rows with entries indexed by $0, 1, \ldots, n$. Equip $\mathbf{R}^{n+1}$ with the bilinear form

$$\langle x, y\rangle = x_0 y_0 - (x_1 y_1 + \ldots x_n y_n), \quad \forall x, y \in \mathbf{R}^{n+1}.$$

The Lorentz group $G = SO_e(1, n)$ is the group of $(n+1) \times (n+1)$ real matrices g such that $\langle xg, yg\rangle = \langle x, y\rangle$, $\forall x, y \in \mathbf{R}^{n+1}$ with $\det(g) = 1$, and with $g_{00} > 0$. We will fix the maximal compact subgroup K to be $G \cap O(n+1)$, so that

$$K = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} \quad : \quad k \in SO(n) \right\}$$

The Cartan involution on G is $\theta(g) = (g^*)^{-1}$. We let A be the subgroup

$$A = \left\{ \mathbf{a}(t) = \begin{pmatrix} \cosh(t) & 0 & \sinh(t) \\ 0 & I_{n-1} & 0 \\ \sinh(t) & 0 & \cosh(t) \end{pmatrix} : t \in \mathbf{R} \right\},$$

and then have

$$M = \left\{ \mathbf{m}(u) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 1 \end{pmatrix} : u \in SO(n-1) \right\},$$

its centralizer in K. The following subgroups are normalized by MA,

$$N = \left\{ \mathbf{n}(x) = \begin{pmatrix} 1 + \frac{1}{2}|x|^2 & x & -\frac{1}{2}|x|^2 \\ x^* & I_{n-1} & -x^* \\ \frac{1}{2}|x|^2 & x & 1 - \frac{1}{2}|x|^2 \end{pmatrix} : x \in \mathbf{R}^{n-1} \right\}$$

and

$$V = \left\{ \mathbf{v}(x) = \theta(\mathbf{n}(x)) = \begin{pmatrix} 1 + \frac{1}{2}|x|^2 & -x & \frac{1}{2}|x|^2 \\ -x^* & I_{n-1} & -x^* \\ -\frac{1}{2}|x|^2 & x & 1 - \frac{1}{2}|x|^2 \end{pmatrix} : x \in \mathbf{R}^{n-1} \right\}.$$

Note that $(1, 0, 1)\mathbf{v}(x) = (1, 0, 1)$ and $(1, 0, -1)\mathbf{n}(x) = (1, 0, -1)$. We define the cone $\mathbb{X}$ to be $\{x \in \mathbf{R}^{n+1} : \langle x, x\rangle = 0 \text{ and } x_0 > 0\}$. This is an orbit of G since $(1, 0, 1)K = \{x : x \in \mathbb{X} \text{ and } x_0 = 1\}$ and for all real numbers t

$$(1, 0, 1)\mathbf{a}(t) = e^t(1, 0, 1) \quad \text{and} \quad (1, 0, -1)\mathbf{a}(t) = e^{-t}(1, 0, -1).$$

The group G has an Iwasawa decomposition ANK and there are functions $H : G \to \mathbf{R}$ and $\kappa : G \to K$ so that $g = \mathbf{a}(H(g))n\kappa(g) \in ANK$ for all $g \in G$. We can find the Iwasawa components of elements of G by considering their action on the points $(1, 0, -1)$ and $(1, 0, 0)$ in $\mathbf{R}^{n+1}$. In particular, for $x \in \mathbf{R}^{n-1}$, we have $\mathbf{v}(x) = \mathbf{a}(t)n\kappa(\mathbf{v}(x))$ in ANK where $e^{-t} = e^{-H(\mathbf{v}(x))} = 1 + |x|^2$ and

$$\kappa(\mathbf{v}(x)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & 2x/(1+|x|^2) & (1-|x|^2)/(1+|x|^2) \end{pmatrix}.$$

2.2. Homogeneous functions. A function $f : \mathbb{X} \to \mathbf{C}$ is said to be homogeneous of degree α if $f(t\,x) = t^{\alpha} f(x) \quad \forall x \in \mathbb{X}, t > 0$. Every homogeneous function is completely determined by its restriction to

$$\Sigma = \{x : x \in \mathbb{X} \text{ and } x_0 = 1\}.$$

The set Σ can be identified with the unit sphere S^{n-1} in $\mathbf{R}^n$ by $(1,\xi) \in \Sigma$ if and only if $\xi \in S^{n-1}$. If f is homogeneous of degree α and $g \in G$ then the function $x \mapsto f(xg)$ is also homogeneous of degree α. We can also consider this as a function of g. If f is homogeneous of degree α on $\mathbb{X}$ and if $g \in G$, $u \in SO(n-1)$, $x \in \mathbf{R}^{n-1}$ and $t \in \mathbf{R}$, then

$$f((1,0,-1)\mathbf{m}(u)\mathbf{a}(t)\mathbf{n}(x)g) = e^{-\alpha t} f((1,0,-1)g). \tag{1}$$

That is, the function $g \mapsto f((1,0,-1)g)$ is left MAN-covariant.

Define a function $c : S^{n-1} \times G \to (0,\infty)$ by setting $c(\xi,g)$ to be the 0th component of $(1,\xi)g$. This follows the device used by Hansen in [9].

For $\xi \in S^{n-1}$ and $g \in G$ define $\xi \cdot g$ to be the element of the sphere S^{n-1} with

$$(1,\xi)g = c(\xi,g)(1,\xi \cdot g).$$

We could write this as

$$(1,\xi)\begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix} = (g_{00} + \xi g_{10}, g_{01} + \xi g_{11})$$

so that $\xi \mapsto \xi \cdot g = (g_{00} + \xi g_{10})^{-1}(g_{01} + \xi g_{11})$ is a fractional linear transformation.

If f is homogeneous of degree α then

$$f((1,\xi)g) = c(\xi,g)^{\alpha} f(1,\xi \cdot g), \quad \forall \xi \in S^{n-1}, g \in G.$$

Let μ denote the normalized rotation invariant measure on S^{n-1}.

LEMMA 1. *If h is a continuous function on the unit sphere S^{n-1} and $g \in G$ then*

$$\int_{S^{n-1}} h(\xi)\, d\mu(\xi) = \int_{S^{n-1}} h(\xi \cdot g) c(\xi,g)^{1-n}\, d\mu(\xi).$$

COROLLARY 1. *If f is a continuous function on $\mathbb{X}$ which is homogeneous of degree $1-n$ then*

$$\int_{S^{n-1}} f(1,\xi)\, d\mu(\xi) = \int_{S^{n-1}} f((1,\xi)g)\, d\mu(\xi), \quad \text{for all } g \in G.$$

2.3. Principal Series. If f_1 is homogeneous of degree α and f_2 is homogeneous of degree β then $f_1 \overline{f_2}$ is homogeneous of degree $\alpha + \overline{\beta}$. Let $\rho = (n-1)/2$.

DEFINITION 1. For each complex number ν let $\mathbf{I}_{\nu}$ denote the space of functions on the cone $\mathbb{X}$ which are:

(1) measurable;
(2) homogeneous of degree $-\rho - \nu$;
(3) are in $L^2(S^{n-1})$ when restricted to the sphere.

The action of G by right translation on elements of $\mathbf{I}_\nu$ is called a *Spherical Principal Series Representation*, with

$$U(\nu, g)f(x) = f(xg), \quad \forall x \in \mathbb{X}, f \in \mathbf{I}_\nu, g \in G.$$

Furthermore, if $\xi \in S^{n-1}$ then $U(\nu, g)f(1, \xi) = c(\xi, g)^{-\rho-\nu} f(1, \xi \cdot g)$. Equation (1) shows that we can identify elements of $\mathbf{I}_\nu$ with functions on G which have the following covariance with respect to MAN,

$$F(m\mathbf{a}(t)ng) = e^{t(\rho+\nu)}F(g), \quad \forall m \in M, t \in \mathbf{R}, n \in N, g \in G.$$

This identification is made by $F(g) = f((1, 0, -1)g), \quad \forall f \in \mathbf{I}_\nu, g \in G$. In this case, the action of $U(\nu, g)$ is right translation by g of a function on G.

If ν is *purely imaginary* and if f_1 and f_2 are in $\mathbf{I}_\nu$, then $f_1\overline{f_2}$ is homogeneous of degree -2ρ, and

$$\int_{S^{n-1}} f_1(1, \xi)\overline{f_2(1, \xi)}\, d\xi = \int_{S^{n-1}} U(\nu, g)f_1(1, \xi)\overline{U(\nu, g)f_2(1, \xi)}\, d\xi$$

That is, the $L^2(S^{n-1})$ inner-product is invariant under this action of G.

PROPOSITION 1. *If ν is purely imaginary, then the principal series $(\mathbf{I}_\nu, U(\nu, \cdot))$ is a unitary representation of G.*

3. Intertwining Operators

PROPOSITION 2. *For each $\nu \in \mathbf{C}$ there is a G-invariant Hermitian pairing between $\mathbf{I}_\nu$ and $\mathbf{I}_{-\overline{\nu}}$ given by*

$$(f_1|f_2) = \int_{S^{n-1}} f_1(1, \xi)\overline{f_2(1, \xi)}\, d\xi.$$

That is, $(U(\nu, g)f_1|U(-\overline{\nu}, g)f_2) = (f_1|f_2), \quad \forall f_1 \in \mathbf{I}_\nu, f_2 \in \mathbf{I}_{-\overline{\nu}}$.

From now on assume that ν is real.

We will construct an operator $A(\nu) : \mathbf{I}_\nu \longrightarrow \mathbf{I}_{-\nu}$ with the intertwining property

$$A(\nu) \circ U(\nu, g) = U(-\nu, g) \circ A(\nu), \quad \forall g \in G,$$

and which is self-adjoint with respect to the $L^2(S^{n-1})$ inner-product.

For such an operator $A(\nu)$, the pairing

$$(f_1, f_2) \mapsto (f_1|A(\nu)f_2)$$

defines a G-invariant Hermitian form on $\mathbf{I}_\nu$.

Consider the function

$$(x, y) \mapsto \langle x, y\rangle^{-\rho+\nu}$$

on $\mathbb{X} \times \mathbb{X}$. In each variable separately this is homogeneous of degree $-\rho + \nu$. If $f \in \mathbf{I}_\nu$ and $y \in \mathbb{X}$ then

$$x \mapsto f(x)\, \langle x, y\rangle^{-\rho+\nu}$$

is homogeneous of degree -2ρ.

DEFINITION 2. For each positive ν let $A(\nu)$ be the operator

$$A(\nu)f(x) = \int_{S^{n-1}} f(1,\xi)\,\langle (1,\xi), x\rangle^{-\rho+\nu}\, d\mu(\xi).$$

It is enough to consider $x \in \Sigma$.

$$A(\nu)f(1,\eta) = \int_{S^{n-1}} f(1,\xi)\,(1-\xi\cdot\eta)^{-\rho+\nu}\, d\mu(\xi).$$

Question: For which values of $\nu > 0$ is this a positive-definite operator on $L^2(S^{n-1})$? Hansen's paper [9] provides an elementary method for answering this question. See also the paper of Faraut [6].

The binomial theorem states that

$$(1-x)^{\nu-\rho} = \sum_{m=0}^{\infty} \binom{\nu-\rho}{m} (-1)^m x^m = \sum_{m=0}^{\infty} \frac{(\rho-\nu)_m}{m!}\, x^m$$

LEMMA 2.

(1) *If* $\varphi \in L^2(S^{n-1})$ *and* $m \geq 0$ *then*

$$\int_{S^{n-1}} \int_{S^{n-1}} \varphi(\xi)\overline{\varphi(\eta)}(\xi\cdot\eta)^m\, d\mu(\xi) d\mu(\eta) \geq 0.$$

(2) *If* $\varphi \in L^2(S^{n-1})$ *and*

$$\int_{S^{n-1}} \varphi(\xi)(\xi\cdot\eta)^m\, d\mu(\xi) = 0$$

for all $m \geq 0$ *and* $\eta \in S^{n-1}$, *then* $\varphi = 0$.

The first part of this lemma is an example from Kreĭn's theory [16] of positive-definite kernels on homogeneous spaces. It follows that if $0 < \nu < \rho$ then $A(\nu)$ is positive-definite.

PROPOSITION 3. *If* $0 < \nu < \rho = (n-1)/2$ *then* $(\mathbf{I}_\nu, U(\nu,\cdot))$ *can be given the structure of a unitary representation of* G *via positive-definite* G*-invariant Hermitian form*

$$(f_1, f_2) \mapsto (f_1 | A(\nu) f_2).$$

This is a well known result, for example see [10,15].

When $\nu - \rho$ is a non-negative integer J,

$$(1-x)^{\nu-\rho} = (1-x)^J = \sum_{m=0}^{J} \binom{J}{m} (-1)^m x^m.$$

In particular, the operator $A(\rho+J)$ has finite-dimensional range in $\mathbf{I}_{-\rho-J}$. Each of the spaces $\mathbf{I}_{\rho+J}$ is reducible, since it contains the G-invariant subspace $\ker A(\rho+J)$.

PROPOSITION 4. *If J is a non-negative integer then* $\ker A(\rho+J)$ *is the subspace*

$$\left\{ f \in \mathbf{I}_{\rho+J} : \int_{S^{n-1}} f(1,\xi)(\xi\cdot\eta)^m \, d\mu(\xi) = 0, \forall m \le J, \eta \in S^{n-1} \right\}$$

PROOF. For each $j \ge 0$ let $\mathcal{H}_j(\mathbf{R}^n)$ denote the space of spherical harmonics of degree j on $\mathbf{R}^n$. Furthermore, let $Z_j(\xi,\eta)$ denote the reproducing kernel for $\mathcal{H}_j(\mathbf{R}^n)$. From Kreĭn [16] we know that $Z_j|_{S^{n-1}\times S^{n-1}}$ is a positive-definite kernel. In fact, it can be expressed in terms of ultraspherical polynomials and there are coefficients $\alpha_{j,\ell}$ so that

$$Z_j(\xi,\eta) = \sum_{0\le\ell\le[j/2]} \alpha_{j,\ell}\,(\xi\cdot\eta)^{j-2\ell}, \quad \forall \xi,\eta \in S^{n-1},$$

with $\alpha_{j,0} \ne 0$. Rearranging this, we have that

$$(\xi\cdot\eta)^j = \sum_{0\le\ell\le[j/2]} \beta_{j,\ell}\, Z_{j-2\ell}(\xi,\eta), \quad \forall \xi,\eta \in S^{n-1},$$

with $\beta_{j,0} \ne 0$. Because the left hand side is a positive-definite kernel on $S^{n-1}\times S^{n-1}$ and because the reproducing kernels are the fundamental $SO(n-1)$-invariant positive-definite kernels, it follows that

$$\beta_{j,\ell} \ge 0, \quad \forall 0 \le \ell \le [j/2].$$

When we look at the kernel $(\xi,\eta) \mapsto (1-\xi\cdot\eta)^J$ we see that it is equal to

$$\sum_{\substack{0\le\ell\le J\\ \ell \text{ even}}} c_{J,\ell}\, Z_\ell(\xi,\eta) \;-\; \sum_{\substack{1\le\ell\le J\\ \ell \text{ odd}}} c_{J,\ell}\, Z_\ell(\xi,\eta)$$

where all the coefficients $c_{J,\ell}$ are positive. For a fixed function f in $L^2(S^{n-1})$ the functions

$$\eta \mapsto \int_{S^{n-1}} f(\xi) Z_j(\xi,\eta)\, d\mu(\xi), \quad j \ge 0,$$

are linearly independent. From this we conclude that $f \in \ker(A(\rho+J))$ if and only if f is orthogonal to $\oplus_{j=0}^{J}\mathcal{H}_j(\mathbf{R}^n)$. □

4. Derived Intertwining Norms

We will show that $\ker A(\rho+J)$ carries a G-invariant positive-definite Hermitian form. It is enough to consider the action of G and $A(\nu)$ on functions whose restriction to the sphere Σ are finite linear combinations of spherical harmonics. Take two such functions in $\ker A(\rho+J)$, say f_1 and f_2. The intertwining property is

$$(f_1|A(\nu)f_2) = (U(\nu,g)f_1|A(\nu)U(\nu,g)f_2)$$

and both sides of this expression are analytic functions of the variable ν. Differentiating with respect to ν gives:

$$(f_1|A'(\nu)f_2) = (U'(\nu,g)f_1|A(\nu)U(\nu,g)f_2) + (U(\nu,g)f_1|A'(\nu)U(\nu,g)f_2) + (U(\nu,g)f_1|A(\nu)U'(\nu,g)f_2).$$

Note that this procedure can be included in the setting of Jantzen filtrations, as defined by Vogan in Def. 3.7 of [27]. See also, [11], section 2 of [24], and section 7 of [28].

PROPOSITION 5. *On the subspace* $\ker A(\rho+J) \subset \mathbf{I}_{\rho+J}$ *the Hermitian form*

$$(f_1, f_2) \mapsto (f_1|A'(\rho+J)f_2)$$

is G-invariant.

Go back to our calculation with the binomial coefficients and isolate the terms containing $(\rho-\nu+J)$. Using Pochhammer notation, we have

$$(\rho-\nu)_m = (\rho-\nu)_J(\rho-\nu+J)(\rho-\nu+J+1)_{m-J-1}$$

provided $m > J$. Letting $\nu \to \rho+J$ we see that

$$\frac{(\rho-\nu)_m}{\rho-\nu+J} \to (-J)_J(1)_{m-J-1} = (-1)^J \times \text{positive term.}$$

THEOREM 1. *If* J *is a non-negative integer then* $(\ker A(\rho+J), U(\rho+J,\cdot))$ *can be equipped with the structure of a unitary representation via*

$$(f_1, f_2) \mapsto (-1)^J(f_1|A'(\rho+J)f_2).$$

Johnson and Wallach [12] did the "dual" case, by considering residues of $A(\nu)$ when ν is $-\rho-J$.

We can view the space $\ker A(\rho+J)$, equipped with this Hilbert space structure, as a space of distributions on S^{n-1} satisfying a moment condition.

If f_1 and f_2 are elements of $\ker A(\rho+J)$ with spherical harmonic expansions $f_\ell(1,\xi) = \sum_{j=J+1}^{\infty} Y_j(f_\ell,\xi)$ on S^{n-1} then there are positive coefficients $\alpha(n,J,j)$ such that

$$(-1)^J(f_1|A'(\rho+J)f_2) = \sum_{j=J+1}^{\infty} \alpha(n,J,j)\,(Y_j(f_1)|\,Y_j(f_2))\,. \tag{2}$$

The coefficients $\alpha(n,J,j)$ can be obtained from calculations in [7, 12].

Fabec [4, 5] used the noncompact picture to describe derived intertwining norms for $SO(1,4)$ and $SU(1,2)$. Other recent work involving derivatives of invariant Hermitian forms has been done by Faraut and Koranyi [8], Arazy [1], and Ørsted and Zhang [21].

4.1. The non-compact picture. Now take elements of $\mathbf{I}_\nu$ to be functions on G with left covariance with respect to MAN, as described by equation (1). When we rewrite the definition of the intertwining operator it becomes

$$A(\nu)f(g) = \int_K \langle (1,0,-1)g, (1,0,-1)k\rangle^{\nu-\rho} f(k)\, dk.$$

Next, restrict g to be an element $\mathbf{v}(z)$ of V, so that

$$(1,0,-1)\mathbf{v}(z) = (1+|z|^2, -2z, |z|^2 - 1), \quad \forall z \in \mathbf{R}^{n-1}.$$

Since all the functions being considered are left-M-invariant, we can transfer the integral over K to an integral over V, see (5.25) in [13]. That is, $A(\nu)f(\mathbf{v}(z))$ is equal to

$$\int_V \langle (1+|z|^2, -2z, |z|^2-1), (1,0,-1)\kappa(\mathbf{v}(x))\rangle^{\nu-\rho} (1+|x|^2)^{-2\rho} f(\kappa(\mathbf{v}(x)))\, dx.$$

But $f(\kappa(\mathbf{v}(x))) = (1+|x|^2)^{\rho+\nu} f(\mathbf{v}(x))$. Hence, we have that

$$A(\nu)f(\mathbf{v}(z)) = 2^{\nu-\rho} \int_{\mathbf{R}^{n-1}} |x-z|^{2\nu-2\rho} f(\mathbf{v}(x))\, dx. \tag{3}$$

In the following material we will be switching back and forth between functions on G and functions on $\mathbf{R}^{n-1}$ and so we use some notation to distinguish the two settings. Take an element f of $\mathbf{I}_\nu$, restrict it to the group V, and then identify it with $f^\flat$ on $\mathbf{R}^{n-1}$ by assigning

$$f^\flat(x) = f(\mathbf{v}(x)), \quad \forall x \in \mathbf{R}^{n-1}.$$

For each $\nu \in \mathbf{C}$ we translate the action of principal series restricted to MAV to an action on functions on $\mathbf{R}^{n-1}$ by

$$(U(\nu, \mathbf{m}(u)\mathbf{a}(t)\mathbf{v}(x))f)^\flat(z) = f(\mathbf{v}(z)\mathbf{m}(u)\mathbf{a}(t)\mathbf{v}(x)) = e^{(\rho+\nu)t} f^\flat(e^t z u + x),$$

for all $u \in SO(n-1)$, $t \in \mathbf{R}$, $x, z \in \mathbf{R}^{n-1}$ and $f \in \mathbf{I}_\nu$. Our hypothesis is that $f|_K$ is in $L^2(M\backslash K)$ and (5.25) in [13] says that this means

$$\int_{\mathbf{R}^{n-1}} |f(\kappa(\mathbf{v}(x)))|^2 e^{2\rho(H(\mathbf{v}(x)))}\, dx < \infty.$$

But $f^\flat(x) = e^{-(\rho+\nu)(H(\mathbf{v}(x)))} f(\kappa(\mathbf{v}(x)))$ and ν is real, so this inequality can be rewritten as

$$\int_{\mathbf{R}^{n-1}} |f((\mathbf{v}(x)))|^2 e^{-2\nu(H(\mathbf{v}(x)))}\, dx = \int_{\mathbf{R}^{n-1}} |f^\flat(x)|^2 (1+|x|^2)^{2\nu}\, dx < \infty. \tag{4}$$

This inequality can also be used to show that elements of $\mathbf{I}_\nu$ are integrable on $\mathbf{R}^{n-1}$ when ν is sufficiently large.

LEMMA 3. *Suppose that J is a non-negative integer. For all $F \in \mathbf{I}_{\rho+J}$ and every polynomial p on $\mathbf{R}^{n-1}$ of degree less than $\rho+2J$ the function $x \mapsto F^\flat(x)p(x)$ is integrable on $\mathbf{R}^{n-1}$.*

PROOF. It suffices to treat a polynomial $p(x)$ which is homogeneous of degree m. Then

$$\int_{\mathbf{R}^{n-1}} \left|p(x)F^\flat(x)\right| \, dx \leq c_p \int_{\mathbf{R}^{n-1}} (1+|x|^2)^{(m/2)} |F^\flat(x)| \, dx$$

and this is bounded by a constant multiple of

$$\left(\int_{\mathbf{R}^{n-1}} |F^\flat(x)|^2 (1+|x|^2)^{2\rho+2J} \, dx\right)^{1/2} \left(\int_{\mathbf{R}^{n-1}} (1+|x|^2)^{m-2\rho-2J} \, dx\right)^{1/2}.$$

The final integral will be finite provided $m < \rho + 2J$. □

In the case when $\nu = 0$ we can identify $\mathbf{I}_0$ with $L^2(\mathbf{R}^{n-1})$ and principal series is unitary. In fact, $U(0, \mathbf{a}(t))f(\mathbf{v}(x)) = e^{\rho t} f^\flat(e^t x)$ is the usual dilation of an element of $L^2(\mathbf{R}^{n-1})$. A Fourier transform argument can be used to show that $\mathbf{I}_0$ is irreducible under the action of $U(0,\cdot)|_{MAV}$ provided $n > 2$, see Theorem 6 on page 151 of [20].

The kernel of the intertwining operator $A(\rho+J)$ is characterized by inequality (4) above and a moment condition.

PROPOSITION 6. *A function f in $\mathbf{I}_{\rho+J}$ belongs to the kernel of $A(\rho+J)$ if and only if*

$$\int_{\mathbf{R}^{n-1}} P_s(z)|z|^{2k} f^\flat(z) \, dz = 0, \quad \forall 0 \leq s+k \leq J, P_s \in \mathcal{H}_s(\mathbf{R}^{n-1}).$$

PROOF. We use Proposition 4 and the formula

$$\int_K F((1,0,-1)k) \, dk = \int_V F((1,0,-1)\kappa(\mathbf{v}(x)))(1+|x|^2)^{-2\rho} \, dx$$

to say that $f \in \mathbf{I}_{\rho+J}$ is an element of $\ker(A(\rho+J))$ if and only if

$$\int_{\mathbf{R}^{n-1}} f^\flat(x)((2x\cdot\eta_1 + (1-|x|^2)\eta_2)^m (1+|x|^2)^{J-m} \, dx = 0, \tag{5}$$

for all $0 \leq m \leq J$, $\eta_1 \in \mathbf{R}^{n-1}$ and $\eta_2 \in \mathbf{R}$. Suppose that $f \in \ker(A(\rho+J))$ and treat the previous integral as a polynomial in η_2, then

$$\int_{\mathbf{R}^{n-1}} f^\flat(x)(x\cdot\eta_1)^j (1-|x|^2)^{m-j} (1+|x|^2)^{J-m} \, dx = 0, \tag{6}$$

for all $0 \leq j \leq m \leq J$ and $\eta_1 \in \mathbf{R}^{n-1}$. But $\ker(A(\rho+J))$ is closed under dilations, so that dilating $f^\flat$ in (6) and changing variables implies that for all $\lambda > 0$

$$\int_{\mathbf{R}^{n-1}} f^\flat(x)(x\cdot\eta_1)^j (\lambda^2-|x|^2)^{m-j} (\lambda^2+|x|^2)^{J-m} \, dx = 0,$$

for all $0 \le j \le m \le J$, $\eta_1 \in \mathbf{R}^{n-1}$. Take the case $m = J$ and treat this as an even polynomial in the real variable λ. We see from the binomial expansion that

$$\int_{\mathbf{R}^{n-1}} f^\flat(x)(x \cdot \eta_1)^j |x|^{2k}\, dx = 0, \qquad \forall 0 \le j + k \le J, \eta_1 \in \mathbf{R}^{n-1}.$$

This implies that

$$\int_{\mathbf{R}^{n-1}} f^\flat(x) \mathcal{Z}_j(x, \eta_1) |x|^{2k}\, dx = 0, \quad \forall 0 \le j + k \le J, \eta_1 \in \mathbf{R}^{n-1},$$

where $\mathcal{Z}_j$ is the reproducing kernel for the space of spherical harmonics of degree j on $\mathbf{R}^{n-1}$. This implies the formula in the statement of the proposition. Conversely, this collection of equations implies (5). □

This tells us that the elements of $\ker(A(\rho + J))$, viewed as functions on $\mathbf{R}^{n-1}$, have zero moments up to order J. The following generalization of Pitt's inequality comes from the proof of Theorem 2 in the paper of Sadosky and Wheeden [23].

PROPOSITION 7. *If J is a nonnegative integer and $f \in \ker(A(\rho + J))$ then*

$$\left(\int_{\mathbf{R}^{n-1}} \left(\frac{|\widehat{f^\flat}(\xi)|}{|\xi|^J} \right)^2 \frac{d\xi}{|\xi|^{2\rho}} \right)^{1/2} \le c_{n,J} \left(\int_{\mathbf{R}^{n-1}} |f^\flat(x)|^2 (1 + |x|^2)^{2J+2\rho}\, dx \right)^{1/2} < \infty.$$

PROOF. In proving theorem 2, on pages 532–3 of [23], Sadosky and Wheeden require the following properties of the function f and the weight w:

(1) that $f(x)p(x)$ be integrable on $\mathbf{R}^n$ for all polynomials p of degree $< k$;
(2) that $\int_{\mathbf{R}^n} f(x)p(x)\, dx = 0$ for all polynomials p of degree $< k$.
(3) that the weight $\widetilde{w}$ be one for which the inequalities (8) on page 523 are valid.

Take $\widetilde{w}(t) = 1$ in Theorem 2 of [23] and set $p = q = 2$, so that $\beta = 2(n-1) - n = n - 2$. The weight w there is $w(t) = |t|^{n-1}$. In this case it follows from the methods used in the proof of the theorem that if f satisfies conditions (1), (2), and (3) above, then

$$\left(\int_{\mathbf{R}^n} \frac{|\widehat{f}(\xi)|^2}{|\xi|^{2k-1+n}}\, d\xi \right)^{1/2} \le c_{n,k} \left(\int_{\mathbf{R}^n} |f(x)|^2 |x|^{2k-1+n}\, dx \right)^{1/2}.$$

If we also assume that $f \in L^2(\mathbf{R}^n)$ then we can say that

$$\begin{aligned}
&\left(\int_{\mathbf{R}^n}\left(\frac{|\widehat{f}(\xi)|}{|\xi|^{k-1}}\right)^2\frac{d\xi}{|\xi|^n}\right)^{1/2}\\
&\quad\le\left(\int_{|\xi|<1}\left(\frac{|\widehat{f}(\xi)|}{|\xi|^{k-1}}\right)^2\frac{d\xi}{|\xi|^n}\right)^{1/2}+\left(\int_{|\xi|>1}\left(\frac{|\widehat{f}(\xi)|}{|\xi|^{k-1}}\right)^2\frac{d\xi}{|\xi|^n}\right)^{1/2}\\
&\quad\le\left(\int_{|\xi|<1}\left(\frac{|\widehat{f}(\xi)|}{|\xi|^{k-1}}\right)^2\frac{|\xi|\,d\xi}{|\xi|^{n+1}}\right)^{1/2}+\left(\int_{|\xi|>1}|\widehat{f}(\xi)|^2\,d\xi\right)^{1/2}\\
&\quad\le c_{n,k}\left(\int_{\mathbf{R}^n}|f(x)|^2|x|^{2k-1+n}\,dx\right)^{1/2}+\|f\|_2.
\end{aligned}$$

Notice that if $2k-1+n\ge 0$ then $|x|^{2k-1+n}=|x|^{2k-2+n+1}\le(1+|x|^2)^{k-1+(n+1)/2}$, for all $x\in\mathbf{R}^n$. Since $n\ge 2$ implies $(n+1)/2\le n$, the previous inequality can then be written as

$$\left(\int_{\mathbf{R}^n}\left(\frac{|\widehat{f}(\xi)|}{|\xi|^{k-1}}\right)^2\frac{d\xi}{|\xi|^n}\right)^{1/2}\le c_{n,k}\left(\int_{\mathbf{R}^n}|f(x)|^2(1+|x|^2)^{k-1+n}\,dx\right)^{1/2}. \tag{7}$$

In the case of $f\in\ker(A(\rho+J))$, use Lemma 3 and Proposition 6 to check conditions (1) and (2) above. We conclude that if $J=k-1$ and we are dealing with functions on $\mathbf{R}^{n-1}$ instead of $\mathbf{R}^n$, then $f\in\ker(A(\rho+J))$ implies the inequality in the statement of the proposition. □

Pitt's inequality and the moment condition (Proposition 6) shows that there is a MAV-equivariant mapping

$$P_J:\ker(A(\rho+J))\to L^2(\mathbf{R}^{n-1})\cong\mathbf{I}_0$$

given by

$$\widehat{(P_JF)}(\xi)=|\xi|^{-\rho-J}\widehat{F}(\xi). \tag{8}$$

It is straightforward to see that P_J satisfies

$$P_J\circ U(\rho+J,g)=U(0,g)\circ P_J,\qquad \forall g\in MAV.$$

From its definition, the operator P_J has trivial kernel. We apply Schur's Lemma to see that P_J is a constant multiple of an isometric isomorphism between the kernel of $A(\rho+J)$, equipped with the derived intertwining norm, and $L^2(\mathbf{R}^{n-1})$, equipped with the usual norm.

THEOREM 2. *Suppose that $n>2$. For each nonnegative integer J there is a positive constant $c_{n,J}$ such that $f\in\ker(A(\rho+J))$ implies that*

$$(-1)^J\,(A'(\rho+J)f|\,f)=c_{n,J}\int_{\mathbf{R}^{n-1}}\left|\widehat{f^\flat}(\xi)\right|^2|\xi|^{-2J-2\rho}\,d\xi.$$

4.2. Comments about radial functions. Suppose we start with a function F on K which is bi-M-invariant, extend it to be an element of $\mathbf{I}_{\rho+J}$, and then restrict it to V. The function on $\mathbf{R}^{n-1}$ obtained in this manner is

$$F^\flat(x) = e^{(\rho+\nu)H(\mathbf{v}(x))}F(\kappa(\mathbf{v}(x))), \quad \forall x \in \mathbf{R}^{n-1}.$$

If F is both left and right M-invariant then $k \mapsto F(k)$ is a function of the entry $k_{n,n}$. But $\kappa(\mathbf{v}(x))_{n,n} = (1-|x|^2)/(1+|x|^2)$.

LEMMA 4. *An element F of $\mathbf{I}_\nu$ has $F|_K$ bi-M-invariant on K if and only if $F^\flat$ is a radial function on $\mathbf{R}^{n-1}$.*

Now assume that $F \in \ker(A(\rho+J))$ and that $F^\flat$ is radial. For every $k \geq 0$, the function $x \mapsto |x|^{2k}F^\flat(x)$ is radial. Lemma 3 says that if $P_s \in \mathcal{H}_s(\mathbf{R}^{n-1})$ with $s > 0$ and $s + 2k < \rho + 2J$ then $x \mapsto P_s(x)|x|^{2k}F^\flat(x)$ is integrable. In addition

$$\int_{\mathbf{R}^{n-1}} P_s(x)|x|^{2k}F^\flat(x)\,dx = 0,$$

since we can change to polar coordinates and use the fact that spherical harmonics of degree greater than zero are orthogonal to constants when integrated over the unit sphere. Every polynomial on $\mathbf{R}^{n-1}$ can be written as a sum of homogeneous polynomials. Every homogeneous polynomial of degree m can be written as a sum of polynomials of the form $P_s(x)|x|^{2k}$ with $s+2k=m$ and P_s a spherical harmonic of degree s.

LEMMA 5. *If $F \in \ker(A(\rho+J))$ has $F^\flat$ radial then*

$$\int_{\mathbf{R}^{n-1}} p(x)F^\flat(x)\,dx = 0$$

for all polynomials p of degree $\leq 2J$.

PROOF. We can assume that p is homogeneous of degree $m \leq 2J$. We can then assume that $p(x) = |x|^{2k}P_s(x)$ for $m = 2k+s$ and $P_s \in \mathcal{H}_s(\mathbf{R}^{n-1})$. If $s > 0$ then the argument above shows that p is orthogonal to $F^\flat$. If $s = 0$ we can use Proposition 6 and the fact that $2k \leq 2J$ to complete the proof. □

Using inequality (7) in the proof of Proposition 7 for a function with zero moments up to degree $2J$, we see that

$$\int_{\mathbf{R}^{n-1}} \left(\frac{|\widehat{F^\flat}(y)|}{|y|^{2J}}\right)^2 \frac{dy}{|y|^{n-1}} \leq c_{n,J}\int_{\mathbf{R}^{n-1}} |F^\flat(x)|^2(1+|x|^2)^{2J+2\rho}\,dx < \infty. \tag{9}$$

Next, apply the operator P_J, defined in (8).

COROLLARY 2. *If J is a integer with $J \geq \rho$ and if $F \in \ker(A(\rho+J))$ has $F^\flat$ radial then*

$$\int_{\mathbf{R}^{n-1}} \left(\frac{|\widehat{P_JF^\flat}(y)|}{|y|^{J-\rho}}\right)^2 \frac{dy}{|y|^{n-1}} \leq c_{n,J}\int_{\mathbf{R}^{n-1}} |F^\flat(x)|^2(1+|x|^2)^{2J+2\rho}\,dx$$

In fact, if ξ is a unit vector in $\mathbf{R}^{n-1}$ then $\int_0^\infty \left|\widehat{P_J F^\flat}(r\xi)\right|^2 r^{-2J+2\rho-1}\,dr < \infty$, *and the value of this integral is independent of the choice of ξ.*

This is the condition on $P_J F^\flat$ required for Caldéron's formula to apply, see section 3.12 of [19].

5. Alternative Realizations

In this section we will show that the representations $(U(\rho+J), \ker(A(\rho+J)))$ can be realized in the kernel of a first order invariant differential operator acting on τ-covariant functions on G. This is similar to the realization of discrete series found by Knapp and Wallach [14]. We also show that although they are not discrete series representations, they exhibit some square-integrability when restricted to MAV.

5.1. First order differential operators. There is a standard construction of a gradient operator ∇. The complexified Lie algebra $\mathfrak{g}_\mathbf{C}$ is identified with right-invariant complex vector fields on G. Let $\mathfrak{p}_\mathbf{C}$ be the -1 eigenspace of θ in $\mathfrak{g}_\mathbf{C}$. For each unitary representation $(\tau, \mathcal{H}_\tau)$ of K, let $C^\infty(G,\tau)$ denote the smooth left τ-covariant functions on G. There is the canonical gradient operator ∇ defined by

$$\nabla f(g)(X) = Xf(g), \quad \forall f \in C^\infty(G,\tau), g \in G, X \in \mathfrak{p}_\mathbf{C}.$$

The value $\nabla f(g)$ is an element of $\mathrm{Hom}_\mathbf{C}(\mathfrak{p}_\mathbf{C}, \mathcal{H}_\tau)$, which is identified with $\mathcal{H}_\tau \otimes \mathfrak{p}_\mathbf{C}^*$. Let $(Ad^*, \mathfrak{p}_\mathbf{C}^*)$ denote the co-adjoint action of K. Then

$$\nabla : C^\infty(G,\tau) \longrightarrow C^\infty(G, \tau \otimes Ad^*)$$

is G-equivariant. Every K-equivariant projection E on $\mathcal{H}_\tau \otimes \mathfrak{p}^*$ gives rise to a G-invariant differential operator $E \circ \nabla$ acting on $C^\infty(G,\tau)$, see [3, 14].

In all that follows we fix an integer $J > 0$ and let $(\tau_{J+1}, \mathcal{H}_{J+1})$ be the action of K on the spherical harmonics $\mathcal{H}_{J+1}(\mathbf{R}^n)$. Since $\mathfrak{p}_\mathbf{C} \cong \mathcal{H}_1(\mathbf{R}^n)$, the tensor product $\mathcal{H}_{J+1}(\mathbf{R}^n) \otimes \mathfrak{p}_\mathbf{C}^*$ includes the K-invariant subspace $\mathcal{H}_{J+2}(\mathbf{R}^n)$ with multiplicity one. Let E_J be the projection on to the orthogonal complement of $\mathcal{H}_{J+2}(\mathbf{R}^n)$ in $\mathcal{H}_{J+1}(\mathbf{R}^n) \otimes \mathfrak{p}_\mathbf{C}^*$. Following [2], the differential operator $\eth_J$ is defined to be

$$\eth_J := E_J \circ \nabla.$$

5.2. Cauchy-Szegö maps. For fixed $J > 0$, as in the previous section, let $R_J : \mathbf{C} \to \mathcal{H}_{J+1}(\mathbf{R}^n)$ be an identification of the complex line with the one-dimensional subspace of M-fixed vectors in $\mathcal{H}_{J+1}(\mathbf{R}^n)$. Take ϕ_J to be the M-fixed unit vector in $\mathcal{H}_{J+1}(\mathbf{R}^n)$ given by $R_J(1)$. We use this to construct Cauchy-Szegö mappings,

$$S_{J,\nu} : \mathbf{I}_\nu \longrightarrow C(G, \tau_{J+1}), \quad \text{where} \quad S_{J,\nu} f(g) = \int_K \tau_{J+1}(k)^{-1} R_J f(kg)\,dk.$$

Such an operator $S_{J,\nu}$ is an example of a τ-quotient map, as defined in [17]. For recent work on general τ-quotient maps for all real-rank one semisimple groups see [26]. The key property is that they do not annihilate the K-type τ_{J+1}, see lemma 3.5.1 in [18].

PROPOSITION 8. *For every $\nu \in \mathbf{C}$ the image of $S_{J,\nu}$ is non-trivial and it contains the K-type $(\tau_{J+1}, \mathcal{H}_{J+1})$ with multiplicity ≥ 1.*

Let $\mathcal{N}_{J,\nu}$ be the image under $S_{J,\nu}$ of the G-invariant subspace of $\mathbf{I}_\nu$ generated by the K-type $(\tau_{J+1}, \mathcal{H}_{J+1})$. For the following result, see [11, 17].

PROPOSITION 9. *(Kunze) For every $\nu \in \mathbf{C}$ the subspaces $\mathcal{N}_{J,\nu}$ and $\mathcal{N}_{J,-\nu}$ are identical in $C(G, \tau_{J+1})$*

Recall from Proposition 4 that the K-type τ_{J+1} occurs in $\ker(A(\rho+J))$ and that $\ker(A(\rho+J))$ is irreducible under the action of G. This means that the $\ker(A(\rho+J))$ has a nontrivial image under $S_{J,\rho+J}$ and that it is equal to the image of $\mathcal{N}_{J,-\rho-J}$. Arguing as in section 5 of [18], the image of $S_{J,-\rho-J}$ is contained in the kernel of the invariant first-order differential operator $\eth_J$ acting on $C^\infty(G, \tau_{J+1})$.

PROPOSITION 10. *For $J > 0$ the Cauchy-Szegö map $S_{J,\rho+J}$ takes $\ker(A(\rho+J))$ into the kernel of the differential operator $\eth_J$ and its image is an irreducible subspace which includes the K-type τ_{J+1}.*

5.3. Main Result. From Johnson and Wallach [12] we know that these representations of G are not square-integrable. However, we claim that for J large enough the image $\mathcal{N}_{J,\rho+J}$ is contained in the subspace

$$\left\{ f \in C^\infty(G, \tau_{J+1}) \,:\, \int_{MAV} |(f(mav)|\phi_J)|^2 \, d(mav) < \infty \right\}.$$

We will use our earlier discussion of both the compact and noncompact pictures of principal series to prove this claim.

The adjoint of the projection $R_J : \mathbf{C} \to \mathcal{H}_{J+1}$ is $R^* : \mathcal{H}_{J+1} \to \mathbf{C}$ with $R_J^*\psi = (\psi|\phi_J)$. When applied to elements of the range of $S_{J,\nu}$, this is

$$R_J^* S_{J,\nu} f(g) = \int_K (\phi_J|\tau_{J+1}(k)\phi_J) f(kg)\, dk = \int_K U(\nu, g) f(k) \overline{(\tau_{J+1}(k)\phi_J|\phi_J)}\, dk.$$

Let us put $\Phi_J(k) = (\tau_{J+1}(k)\phi_J|\phi_J)$ for all $k \in K$. In the $L^2(K)$ inner product we have

$$R_J^* S_{J,\nu} f(g) = (U(\nu, g) f \,|\, \Phi_J)\,.$$

Fix $\nu = \rho + J$ and extend Φ_J to be an element of $\mathbf{I}_{\rho+J}$. The function Φ_J is an element of $\ker(A(\rho+J))$, it belongs to the K-type τ_{J+1} in that space, and our earlier discussion shows that there is a nonzero constant α_J such that

$$(-1)^J (A'(\rho+J)\Phi_J \,|\, F) = \alpha_J (\Phi_J \,|\, F), \quad \forall F \in \ker(A(\rho+J)).$$

This means that

$$R_J^* S_{J,\rho+J} f(g) = \alpha_J^{-1} \left(U(\rho+J,g)f \,|\, A'(\rho+J)\,\Phi_J\right), \forall f \in \ker(A(\rho+J)), g \in G.$$

We are interested in the behaviour of $R_J^* S_{J,\rho+J} f$ on MAV. The function Φ_J is left and right M-invariant on K and so the function $x \mapsto \Phi_J(\kappa(\mathbf{v}(x)))$ is radial on V.

In our calculations with the operator P_J we saw that if f_1 and f_2 are in the kernel of $A(\rho+J)$ then

$$(-1)^J \left(A'(\rho+J)f_1 \,|\, f_2\right) = c \int_V P_J f_1^\flat(x) \overline{P_J f_2^\flat(x)}\, dx.$$

Furthermore, P_J intertwines $U(\rho+J,g)$ and $U(0,g)$ for all $g \in MAV$. From this it follows that $f \in \ker(A(\rho+J))$ implies

$$R_J^* S_{J,\rho+J} f(mav) = c \int_V \left(U(0,mav)P_J f^\flat\right)(x) \overline{P_J \Phi_J^\flat(x)}\, dx.$$

The image of $R^* S_{J,\rho+J}$ consists of matrix coefficients of the unitary representation $U(0,\cdot)|_{MAV}$ acting on $L^2(V)$. The right Haar measure on MAV is given by

$$\int_{MAV} F(mav)\, d(mav) = \int_M \int_{\mathbf{R}} \int_{\mathbf{R}^{n-1}} F(m\mathbf{a}(t)\mathbf{v}(x)) e^{-2\rho t}\, dx\, dt\, dm.$$

Let Q be the function $\overline{P_J \Phi_J^\flat}$ on $\mathbf{R}^{n-1}$.

In the equation above, $U(0,\mathbf{m}(u)\mathbf{a}(t)\mathbf{v}(z))F(x) = e^{\rho t} F(e^t x u + z)$, and

$$\int_{\mathbf{R}^{n-1}} e^{\rho t} F(e^t x m + z) Q(x)\, dx = e^{-\rho t} \int_{\mathbf{R}^{n-1}} F(y+z) Q(e^{-t} y)\, dy = e^{-\rho t} F * Q_t(z),$$

with $Q_t(x) = Q(e^{-t}x)$. The Fourier transform of this is $\widehat{Q_t}(\xi) = e^{2\rho t} \widehat{Q}(e^t \xi)$. The square of the $L^2(\mathbf{R}^{n-1} \times \mathbf{R}, e^{-2\rho t}\, dz dt)$ norm of $e^{-\rho t} F * Q_t(z)$ is

$$\int_{\mathbf{R}} e^{-4\rho t} \|F * Q_t\|_2^2\, dt = \int_{\mathbf{R}} \int_{\mathbf{R}^{n-1}} \left|\widehat{F}(\xi)\right|^2 \left|\widehat{Q}(e^t \xi)\right|^2 d\xi\, dt$$

Recall that Q is radial. For a given $\xi \neq 0$, let $e^r = |\xi|$, so that

$$\int_{\mathbf{R}} \left|\widehat{Q}(e^t \xi)\right|^2 dt = \int_{\mathbf{R}} \left|\widehat{Q}\left(e^{t+r} \frac{\xi}{|\xi|}\right)\right|^2 dt.$$

But integration over the line is translation invariant and so we need only estimate $\int_{\mathbf{R}} \left|\widehat{Q}(e^t \xi)\right|^2 dt$ when ξ is a unit vector. Now take $|\xi| = 1$ and let $s = e^t$,

$$\int_{\mathbf{R}} \left|\widehat{Q}(e^t \xi)\right|^2 dt = \int_0^\infty \left|\widehat{Q}(s\xi)\right|^2 \frac{ds}{s}. \tag{10}$$

This last expression is independent of the choice of unit vector ξ.

In the discussion about moments of radial elements of $\ker(A(\rho+J))$ Pitt's inequality showed that the integral (10) is finite whenever $J \geq \rho$.

THEOREM 3. *(Main Result) If J is an integer with $J \geq \rho = (n-1)/2$ and $n > 2$, then the Cauchy-Szegö mapping $S_{J,\rho+J}$ takes* $\ker(A(\rho+J))$ *into the space of τ_{J+1}-equivariant functions F on G with the properties that*

$$\int_{MAV} |R^*F(mav)|^2 \, d(mav) < \infty \quad \text{and} \quad \eth_J F = 0.$$

In fact, there is a constant $c > 0$ with

$$\int_{MAV} |R^* S_{J,\rho+J} F(mav)|^2 \, d(mav) = c(-1)^J \, (A'(\rho+J)f \,|\, f)\,,$$

for all $f \in \ker(A(\rho+J))$.

PROOF. We have done almost all of this already. The last identity is another argument based on Schur's lemma, since the $L^2(MAV)$ norm is a right-MAV-invariant Hermitian form on a space which we have already seen to be irreducible under MAV. □

Notice that this says that $f \mapsto \|R_J^* S_{J,\rho+J} f\|_{L^2(MAV)}$ is a G-invariant norm on the irreducible space $\ker(A(\rho+J))$.

5.4. Affine symmetric spaces. Schlichtkrull [25] has shown that there is a G-equivariant mapping from $\ker(A(\rho+J))$ to an L^2-eigenspace of the Laplace operator on the non-Riemannian symmetric space $SO_e(1,n-1)\backslash SO_e(1,n)$. See also [7, 22].

REFERENCES

[1] J. Arazy, *Realization of the invariant inner products on the highest quotients of the composition series*, Arkiv för mat. **30** (1992), 1–24.

[2] K. M. Davis, J. E. Gilbert, and R. A. Kunze, *Invariant differential operators in harmonic analysis on real hyperbolic space*, Proc. Centre for Math. Analysis **16** (1988), 79–91.

[3] K. M. Davis, J. E. Gilbert, and R. A. Kunze, *Elliptic differential operators in harmonic analysis, I. Generalized Cauchy-Riemann systems*, Amer. J. Math. **113** (1991), 75–116.

[4] R. C. Fabec, *Localizable representations of the de Sitter group*, J. d'Analyse Math. **35** (1979), 151–208.

[5] R. C. Fabec, *Homogeneous distributions on the Heisenberg group and representations of $SU(2,1)$*, Trans. AMS **328** (1991), 351–391.

[6] J. Faraut, *Noyaux spheriques sur un hyperboloide a une nappe*, Lecture Notes in Math. **497** (1975), 172–210.

[7] ———, *Distributions sphériques sur les espaces hyperboliques*, J. Math. pures et appl. **58** (1979), 369–444.

[8] ———and A. Koranyi, *Function spaces and reproducing kernels of bounded symmetric domains*, J. Funct. Analysis **88** (1990), 64–89.

[9] M. L. Hansen, *Weak amenability of the universal covering group of $SU(1,n)$*, Math. Ann. **288** (1990), 445–472.

[10] R. E. Howe and E.-C. Tan, *Homogeneous functions on light cones: the infinitesimal structure of some degenerate principal series representations*, Bull. AMS **28** (1993), 1–74.

[11] J. E. Gilbert, R. A. Kunze, and C. Meaney, *Derived intertwining norms for reducible spherical principal series*, to appear, J. Austral. Math. Soc..

[12] K. D. Johnson and N. R. Wallach, *Composition series and intertwining operators for the spherical principal series. I*, Trans. AMS **229** (1977), 137–173.

[13] A. W. Knapp, *Representation Theory of Semisimple Groups*, Princeton Univ. Press, Princeton, NJ, 1986.

[14] _____ and N. R. Wallach, *Szegö kernels associated with discrete series*, Invent. Math. **34** (1976), 163–200.
[15] B. Kostant, *On the existence and irreducibility of certain series of representations*, Bull. AMS **75** (1969), 627-642.
[16] M. G. Kreĭn, *Hermitian-positive kernels on homogeneous spaces, I and II*, Ukrain. Mat. Ž. **1** (1949), 64–98; Translated in: *AMS Translations*, Series 2, **34** (1963) 69–164 **2** (1950), 10–59.
[17] R. A. Kunze, *Quotient representations*, Turin and Milan, Topics in Modern Harmonic Analysis: **1** (1983), 57–80.
[18] C. Meaney, *Cauchy-Szegö maps, invariant differential operators and some representations of $SU(n+1,1)$*, Trans. AMS **313** (1989), 161–186.
[19] Y. Meyer, *Wavelets and Operators*, Cambridge Univ. Press, Cambridge, 1992.
[20] M. A. Naimark, *Linear Representations of the Lorentz Group*, Pergamon Press, Oxford, 1964.
[21] B. Ørsted and G. Zhang, *Reproducing kernels and composition series for spaces of vector-valued holomorphic functions on tube domains*, preprint, June 1993.
[22] R. Raczka, N. Limić, and J. Niederle, *Discrete degenerate representations of noncompact rotation groups. I*, J. Math. Phys. **7** (1966), 1861–1876.
[23] C. Sadosky and R. Wheeden, *Some weighted norm inequalities for the Fourier transform of functions with vanishing moments*, Trans. AMS **300** (1987), 521–533.
[24] S. Sahi, *Unitary representations on the Shilov boundary of a symmetric tube domain*, Contemp. Math. **145** (1993), 275–286.
[25] H. Schlichtkrull, *Eigenspaces of the Laplacian on hyperbolic spaces; composition series and integral transforms*, J. Funct. Analysis **70** (1987), 194–219.
[26] H. van der Ven, *Vector-valued Poisson transforms on Riemannian symmetric spaces of rank one*, J. Funct. Analysis **119** (1994), 358–400.
[27] D. Vogan, *Unitarizability of certain series of representations*, Ann. Math. **120** (1984), 141–187.
[27] D. Vogan, *The unitary dual of G_2*, Invent. Math. **116** (1994), 677–791.

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THE HOLOMORPHIC CAUCHY–SZEGÖ KERNEL FOR NONHOLOMORPHIC DISCRETE SERIES REPRESENTATIONS OF $SU(2,1)$

SIMON G. GINDIKIN

Dedicated to Ray A. Kunze

ABSTRACT. We consider some problems of integral geometry on the manifold of flags in $\mathbb{C}P^n$ and discuss their connections with complex analysis and the theory of representations.

I am happy to be among friends and colleagues of Ray Kunze who can pay tribute to this remarkable mathematician. I believe that Ray and I are from the generation in the representation theory for whom explicit formulas have the highest priority. It is the reason why I chose to talk about some very simple and explicit formulas in the conference in his honor.

I will discuss the analogue of Cauchy-Szegö operator for discrete series – the intertwining operator from nonunitary principal series into a discrete series representation. This fundamental construction is due to Knapp and Wallach [1, 2]. For $SL(2,\mathbb{R})$ the kernel function is just the Cauchy kernel for derivatives of holomorphic functions. My aim is to relate this operator (in the multidimensional case) with complex (moreover, holomorphic) analysis. It is another step in the general program of holomorphic realizations of nonholomorphic discrete series [3]. On the other hand, the classical Cauchy kernel should not be tied too closely with $SL(2,\mathbb{R})$: its tie lies rather with the complex line (affine or projective), the construction should make sense for any domain in there and it is only for the special case of the circle that the group interpretation does arise. We want to understand which structures define the Cauchy-Szegö kernel, with an eye on generalization to other groups. It turns out that complex groups are the natural setting that provides the essential structures (indeed, more generally, non-group structures also give the natural setting).

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1. Geometrical Notations

We will consider the simplest example - the group $SU(2,1)$. This group acts on the projective plane $\mathbb{C}P^2$ with the homogeneous coordinates $z = (z_0, z_1, z_2)$ and preserves the domains

$$D_\pm = \{z \in \mathbb{C}P^2 : |z_0|^2 - |z_1|^2 - |z_2|^2 \gtrless 0\}. \tag{1}$$

Let us emphasize that practically all our constructions will deal with the projective geometry of $\mathbb{C}P^2$, rather the domains (1). (In other words, the group $SL(3,\mathbb{C})$, rather than $SU(2,1)$, is involved.) More specifically, instead of the domain D_+ in (1), we can consider any convex domain $D_+ \subset \mathbb{C}P^2_z$ with (for simplicity's sake) smooth boundary.

Let $\mathbb{C}P^2_Z = (\mathbb{C}P^2_z)'$ be the dual projective plane

$$\langle Z, z\rangle = Z^i z_i = Z^0 z_0 + Z^1 z_1 + Z^2 z_2. \tag{2}$$

Here, we identify $Z \in \mathbb{C}P^2_Z$ with the line defined by $\langle Z, \cdot\rangle = 0$ in $\mathbb{C}P^2_z$. Let $(D')_+ \subset \mathbb{C}P^2_Z$ be the set of lines Z which are contained in D_-; then $(D')_+$ will be the convex domain in $\mathbb{C}P^2_Z$ (dual to D_-), whereas $(D')_-$ will correspond to the lines in $\mathbb{C}P^2_z$ which intersect D_+.

Let F be the manifold of flags in $\mathbb{C}P^2$:

$$F = \{(z, Z), z \in \mathbb{C}P^2_z, Z \in \mathbb{C}P^2_Z, \langle Z, z\rangle = 0\}, \quad \dim F = 3, \tag{3}$$

and $\mathcal{D}$ the domain of those flags (z, Z) with $z \in D_-$ and the line Z intersects D_+:

$$\mathcal{D} = \{(z, Z) \in F, z \in D_-, Z \in (D')_-\}. \tag{4}$$

If B is the (common) boundary of the domains $D_\pm$, let $\mathcal{B}$ be the set of flags such that $z \in B$ and Z is the tangent line to B at the point z. Hence, $\mathcal{B}$ is a (real) three-dimensional manifold in F which is a part of the boundary of the domain $\mathcal{D}$. It is totally real and, in a certain sense, the Shilov boundary for $\mathcal{D}$. The manifold $\mathcal{B}$ is isomorphic as a real manifold to the boundary B, but B has CR structure and the map $B \to \mathcal{B}$ destroys this structure ($\mathcal{B}$ is totally real but a memory about CR structure on B "lives" in the way that $\mathcal{B}$ is embedded in F).

2. The Cauchy-Szegö Kernel (first version)

Nonunitary principal series are realized in the sections of line bundles on $\mathcal{B}$. They can be thought of as homogeneous functions satisfying

$$f(\lambda z, \mu Z) = \lambda^\alpha \mu^\beta f(z, Z), \quad f \in \Gamma(\vartheta(\alpha, \beta)). \tag{5}$$

We will consider only the simplest case when $\alpha = \beta = -1$ and will denote this space of (smooth) section as $\mathcal{S}(\mathcal{B})$. The corresponding kernel is analogous to the classical Cauchy kernel for functions rather than for derivatives.

Let us define the standard "holomorphic" volume element on F:

$$\Omega(z, Z) = \frac{[z, dz, dz] \wedge [U, Z, dZ]}{\langle U, z\rangle}, \qquad \langle Z, z\rangle = 0, \ \langle U, z\rangle \neq 0. \tag{6}$$

Observe that in the determinant

$$[a_1, a_2, a_3] = \det(a_1, a_2, a_3)$$

if some columns a_j are forms we use the exterior product for the computation of the determinant. The form Ω is quasi-invariant under the projective group $SL(3,\mathbb{C})$ and independent of U when Ω is restricted to F (but not on the $\mathbb{C}P_z^2 \times \mathbb{C}P_Z^2$). Sometimes, it is convenient to define the line Z which goes through z by introducing another point ν on the line. Then

$$\Omega(z,\nu) = [z, dz, dz] \wedge [z, \nu, d\nu].$$

Remark. The formula (6) looks asymmetric between z and Z. But the origin of (6) is explained by the formula

$$\Omega(z,Z) = \operatorname*{Res}_{\langle Z,z\rangle=0} \frac{[z,dz,dz] \wedge [Z,dZ,dZ]}{\langle Z,z\rangle}$$

and the asymmetry and usage of an element U from which Ω is independent are the result of the computation of the residue on F; we can interchange Z, z and will obtain the same form on F. More precisely, we will obtain the form which will be different on $\mathbb{C}P_z^2 \times \mathbb{C}_Z^2$ but will coincide with (6) when restricted to F.

Now we are ready to describe the first version of the Cauchy-Szegö operator:

$$Sf(w,W) = \int_{\mathcal{B}} \frac{f(z,Z)\Omega(z,Z)}{\langle Z,w\rangle\langle z,W\rangle}, \qquad (w,W) \in D_+ \times (D')_+. \tag{7}$$

This operator, acting on the sections $f \in \mathcal{S}(\mathcal{B})$, yields sections of the holomorphic line bundles $\vartheta(-1,1)$ on $D_+ \times (D')_+$. Note that f depends on 3 variables and Sf on 4 variables. So we expect that the image must satisfy an additional condition. Indeed, we have

$$\begin{aligned} &\Box(Sf) = 0, \\ &\Box = \frac{\partial^2}{\partial w_i \partial W^i} = \frac{\partial^2}{\partial w_0 \partial W^0} + \frac{\partial^2}{\partial w_1 \partial W^1} + \frac{\partial^2}{\partial w_2 \partial W^2}. \end{aligned} \tag{8}$$

Remark. Let us remark for comparison that the classical Cauchy formula in projective form (on $\mathbb{C}P^1$) is the operator on sections of the line bundle $\vartheta(-1)$ with the kernel

$$\frac{[z,dz]}{\langle Z,z\rangle}, z = (z_0,z_1), Z = (Z^0,Z^1)$$

We will denote the space of (holomorphic) sections of line bundle $\vartheta(-1,-1)$ on $D_+ \times (D')_+$ satisfying the equation $\Box g = 0$ as $\mathcal{A}(D_+ \times (D')_+)$. So S *is the operator from* $\mathcal{S}(\mathcal{B})$ *into* $\mathcal{A}(D_+ \times (D')_+)$.

To prove (8) it is enough to differentiate directly (7) and to use the condition $\langle z, Z\rangle = 0$.

Let us emphasize that the construction of (7) depends only on the projective geometry and makes sense for any convex domain D_+. Also, we can realize $D_+ \times (D')_+$ as the domain in the homogeneous complex (Stein) manifold $M = SL(3,\mathbb{C})/GL(2,\mathbb{C})$.

We can say a little bit more about the homogeneous domain D_+ in (1). The operator S is an intertwining operator and we have the representation of $SU(2,1)$ on the space $\mathcal{A}(D_+ \times (D')_+)$.

Inside $D_+ \times (D')_+$ there is the real submanifold

$$\Lambda = \{(z,Z) : z = (z_0, z_1, z_2), Z = (\bar{z}_0, -\bar{z}_1, -\bar{z}_2), z \in D_+\}. \tag{9}$$

The restriction of the operator $\Box$ *on* Λ *will be elliptic and all solutions of* $\Box g = 0$ *on* Λ *admit holomorphic extensions on* $D_+ \times (D')_+$. The operator $\Box\big|_\Lambda$ is the analogue of the Schmid operator for this case and if we restrict (7) to those $(w,W) \in \Lambda$ we will obtain the Szegö operator of Knapp-Wallach [2].

3. The Geometry of the Operator $\Box$

The operator S sends any object subsisting on $\mathcal{B}$ (which is the essential part of the boundary of the domain $\mathcal{D} \subset F$) to an object on the domain $D_+ \times (D')_+ \subset M$. What is the relationship between $D_+ \times (D')_+$ and $\mathcal{D}$? We can think of $(w,W) \in D_+ \times (D')_+$ as rational curves $E_{w,W}$ in the domain $\mathcal{D}$:

$$E_{w,W} = \{(z,Z) \in F : \langle Z, w\rangle = \langle z, W\rangle = 0\}, \quad E_{w,W} \cong \mathbb{C}P^1. \tag{10}$$

Using this interpretation we can give a geometrical interpretation of the operator $\Box$. For each curve $E_{w,W}$, consider the isotropic cone $\mathcal{V}_{w,W} \subset M$ formed by those $(z,Z) \in M$ so that the curves $E_{w,W}$ and $E_{z,Z}$ are intersecting. Let $V_{w,W}$ be the infinitesimal image of this cone in the tangent space $T_{w,W}M$. We obtain a (complex) conformal structure on $M \supset D_+ \times (D')_+$ (a version of Plücker construction). Then $\Box$ is just the Laplacian of this structure. The restriction of this conformal structure on the real submanifold $\Lambda \subset M$ will have the Riemann signature. It corresponds to the fact that the corresponding curves $E_{w,W}, (w,W) \in \Lambda$, are the fibers of the (real) fibering of $\mathcal{D} \subset F$ over D_+ (i.e., these curves do not intersect). Of course, the existence of such a fibering (an analogue of Hopf fibering) holds only for homogeneous domain (1).

On M we will use coordinates different from the Stiefel coordinates (w,W); namely, we will pin down (w,W) by a pair of flags $(u,U),(v,V)$ from the curve $E_{w,W}$ with the extra condition

$$\langle w,U\rangle = \langle w,V\rangle = \langle W,u\rangle = \langle W,v\rangle = \langle u,U\rangle = \langle v,V\rangle = \langle u,V\rangle + \langle v,U\rangle = 0 \tag{11}$$

Then the operator (7) has the form

$$S'f(u,U|v,V) = \int_{\mathcal{B}} \frac{f(z,Z)\Omega(z,Z)}{P[u,v,z][U,V,Z]}$$

We will use $(w,W),(u,U),(v,V)$ simultaneously in various combinations.

Remark. Of course, we can represent the components of $w(W)$ as the minors of the matrices (U,V) and (u,v) respectively but the simultaneous use of all these coordinates makes the formulas more compact.

For example we can rewrite the operator $\Box$ as

$$\Box' = (w\frac{\partial}{\partial u})(W\frac{\partial}{\partial V}) - (w\frac{\partial}{\partial v})(W\frac{\partial}{\partial U}) \tag{12}$$

where $w\frac{\partial}{\partial u} = w_0\frac{\partial}{\partial u_0} + w_1\frac{\partial}{\partial u_1} + w_2\frac{\partial}{\partial u_2}$, etc. Using the identity

$$w\frac{\partial}{\partial u} = V\frac{\partial}{\partial W}$$

we can also write $\Box$ as

$$\Box'' = (V\frac{\partial}{\partial W})(u\frac{\partial}{\partial w}) - (v\frac{\partial}{\partial w})(U\frac{\partial}{\partial W})$$

These forms may seem more complicated than (7) but they uncover a structure which is important for applications.

4. Holomorphic cohomology and the second version of the Cauchy-Szegö formula

The operator S relates the boundary of one complex domain $\mathcal{D}$ to another complex domain $D_+ \times (D')_+$. This situation is not quite usual for boundary value problems in complex analysis. We would like to have constructions more closely related to $\mathcal{D}$. The problem is that the domain $\mathcal{D}$ is not a Stein manifold (the domain $D_+ \times (D')_+$ is) and has so far no holomorphic objects there. This is the reason why we need to consider nonholomorphic discrete series.

Let us give to the formula (7) the following interpretation. Let $\mathcal{D}$ be a manifold of those pairs of flags $\{(u,U),(v,V)\}$ from $\mathcal{D}$ so that there is a curve $E_{w,W}, (w,W) \in D_+ \times (D')_+$, which contain both of the flags. In other words, it means that the point w is the intersection of the lines U and V (and must lie in D_+) and the line W goes through the points u, v (and must lie inside D_-, i.e., $W \in (D')_+$). The manifold $\tilde{\mathcal{D}}$ is already a Stein manifold and we have the canonical covering

$$\pi : \tilde{\mathcal{D}} \to \mathcal{D}, \pi((u,U),(v,V)) = (u,U). \tag{13}$$

The manifold $\mathcal{D}$ is 1-pseudoconcave and instead of holomorphic functions we can consider a 1-dimensional $\bar{\partial}$-cohomology there (it is the form in which nonholomorphic discrete series are realized). Due to the existence of the Stein covering (13), we can follow [3] and describe such a cohomology in the holomorphic language.

Namely, if $g \in \mathcal{A}(D_+ \times (D')_+)$ let us consider the 1-form

$$\kappa g = \frac{1}{\langle w,W\rangle}\Big\{(w\frac{\partial}{\partial w})g\cdot\langle W,dv\rangle + (W\frac{\partial}{\partial V})g\cdot\langle w,dV\rangle + g\cdot(\langle V,dv\rangle - \langle v,dV\rangle)\Big\} \tag{14}$$

(where we mix coordinates u,U,v,V,w,W). We have holomorphic forms on the manifold $\mathcal{D}$ with differentials along the fibers of $\pi\colon \tilde{\mathcal{D}} \to \mathcal{D}$ only.

The equation $\Box g = 0$ is equivalent to the condition that the form κg is closed along the fibers of π: it is closed as a form on (v, V) for any fixed $(u, U) \in \mathcal{D}$. So we have closed holomorphic 1-forms with parameters $\phi(v, U \mid v, V, dv, dV)$. The forms κg satisfy an extra condition:

(H) *ϕ, as a function from the parameters (u, U), is constant along the curve $E_{w,W}$ connecting $(u, U), (v, V)$: $\phi = \phi(w, W; dv, dV)$.*

We can consider the cohomology for such complexes of holomorphic forms and the condition (H) picks up then the unique representative in each cohomology class (holomorphic Hodge theorem [3]).

We will denote the space of holomorphic forms $\phi(w, W; dv, dV)$ satisfying the condition (H) as $\mathcal{L}(\mathcal{D})$. In this way, in the homogeneous case we have the realization of the representation of $SU(2,1)$ as the space $\mathcal{L}(\tilde{\mathcal{D}})$ and κ is the intertwining operator from $\mathcal{A}(D_+ \times (D')_+)$ to $\mathcal{L}(\tilde{\mathcal{D}})$.

The composition $\kappa \circ S$ gives us the second version of Cauchy-Szegö operator which sends sections on $\mathcal{B}$ to forms on $\mathcal{L}(\tilde{\mathcal{D}})$:

(15)

$$(\kappa \circ S) f(w, W; dv, dV) = \frac{1}{\langle w, W\rangle} \int_{\mathcal{B}} \frac{f(z, Z)\Omega(z, Z)}{\langle w, Z\rangle^2 \langle W, z\rangle^2} \{\langle V, z\rangle \langle w, Z\rangle \langle W, dv\rangle + \langle v, Z\rangle \langle W, z\rangle \langle w, dV\rangle + \langle w, Z\rangle (\langle V, dv\rangle - \langle v, dV\rangle)\},$$

$(w, W) \in D_+ \times (D')_+, \langle W, v\rangle = \langle V, w\rangle = 0$. There is a very simple operator from $\mathcal{L}(\tilde{\mathcal{D}})$ in the Dolbeault cohomology $H^{(1)}(\mathcal{D}) = H^{(1)}(\mathcal{D}, \vartheta(-1, -1))$. Let Γ be a section (of course, nonholomorphic) of the fibering $\pi\colon \tilde{\mathcal{D}} \to \mathcal{D}$, $(v, V) = \gamma(u, U)$. Let us consider the operator

(16)
$$\mathcal{R}\colon \mathcal{L}(\tilde{\mathcal{D}}) \to H^{(1)}(\mathcal{D}),$$
$$\phi \in \mathcal{L}(\tilde{\mathcal{D}}) \mapsto \{\phi|\Gamma\}^{(0,1)}$$

where we push down ϕ to $\mathcal{D}$ via the pullback through Γ and then we take the $(0,1)$-part of the form obtained. This operator gives a $\bar{\partial}$-closed $(0,1)$-form on $\mathcal{D}$ and we descend to the cohomology level, we will obtain an operator $\mathcal{R} : \mathcal{L}(\tilde{\mathcal{D}}) \to H^{(1)}(\mathcal{D})$ which is independent from the choice of the section. This gives an isomorphism between these spaces. The composition of the operators (15) and (16) gives a new version of the Cauchy-Szegö operator:

$$\mathcal{R} \circ \kappa \circ S : \mathcal{S}(\mathcal{B}) \to H^{(1)}(\mathcal{D}).$$

Remark. In the homogeneous case (1) it is convenient to use the fibration Λ of $\mathcal{D}$ over D_+ to get an explicit construction of the section Γ.

Lastly, we will construct the intertwining operators which maps the Dolbeault cohomology to the solution space of $\Box g = 0$. This is accomplished through the Penrose operator $\mathcal{P}$, which is essentially just the integration of the $\bar{\partial}$-cohomology in $\mathcal{D}$ over curves $E_{w,W}$. If Ψ is a $\bar{\partial}$-closed $(0,1)$-form on $\mathcal{D}$ with coefficients in $\vartheta(-1, -1)$, then

(17)
$$\mathcal{P}\Psi(w, W) = \int_{\mathbb{C}P^1_+} \Psi\big|_{z=\tau_0 u+\tau_1 v,\, Z=\tau_0 U+\tau_1 V} (-\tau_1 d\tau_1 + \tau_1 d\tau_0),$$

where $\langle w, W\rangle \in D_+ \times (D')_+$, $\langle u, v\rangle \in W$, $w = U \cap V$, $\langle w, V\rangle = \langle v, V\rangle = \langle u, V\rangle - \langle v, U\rangle = 0$. It is easy to check directly that $\mathcal{P}\Psi \in \mathcal{A}(D_+ \times (D')_+)$ and that $\Box(\mathcal{P}\Psi) = 0$. As a result, we have the following diagram relating four spaces

$$
\begin{array}{ccccc}
 & & \mathcal{A}(D_+ \times (D')_+) & \xrightarrow{\kappa} & \mathcal{L}(\tilde{\mathcal{D}}) \\
 & \overset{S}{\nearrow} & & & \downarrow \mathcal{R} \\
\mathcal{S}(\mathcal{B}) & & & \overset{\mathcal{P}}{\nwarrow} & \\
 & & & & H^{(1)}(\mathcal{D})
\end{array}
\tag{18}
$$

This diagram is commutative up to a choice of the normalized factors.

5. Integral geometry on F

The Penrose transform $\mathcal{P}$ (17) is related to the problem of integral geometry for $\bar{\partial}$-cohomology in the domain $\mathcal{D} \subset F$. Now we will consider the problem of integral geometry on the whole manifold F. We will consider the space $\mathcal{S}(F)$ of homogeneous functions

$$
f(\lambda z, \mu Z) = \lambda^{-1}\mu^{-1} f(zZ), \quad \langle z, Z\rangle = 0, \tag{19}
$$

which are C^∞ (therefore, real functions) and we will integrate them along rational curves $E_{w,W}$, $(w, W) \in M$. In other words,

(20)

$$
\begin{aligned}
\hat{f}(u, U|v, V) &= \int_{\mathbb{C}P^1_\tau} f(\tau_0 u + \tau_1 v, \tau_0 U + \tau_1 V)(-\tau_0 d\tau_1 + \tau_1 d\tau_0) \wedge (-\bar{\tau}_1 d\bar{\tau}_1 + \bar{\tau}_1 d\bar{\tau}_0) \\
&= \int_F f(z, Z)\delta([z, w, v])\delta([Z, U, V])\Omega(z, Z) \wedge \overline{\Omega(z, Z)},
\end{aligned}
$$

where $\langle u, U\rangle = \langle v, U\rangle = \langle u, V\rangle + \langle v, V\rangle = 0$. The problem of integral geometry is to reconstruct f from $\hat{f}$. It is a over-determined situation and we will describe the solution using the κ-trick of Gelfand-Graev-Shapiro. Let κ be the operator (14) and $\bar{\kappa}$ be the complex conjugate operator. Then for arbitrary $(u, U) \in F$ the form $(\kappa \wedge \bar{\kappa})\hat{f}$ is the closed $(1,1)$-form on $(v, V) \in F_{u,U}$ for the arbitrary $(v, V) \in F$ (we require $u \neq \lambda v, U \neq \mu V$). It is commonplace to expect the following (in fact, true) relationship

$$
\int_\gamma (\kappa \wedge \bar{\kappa})\hat{f} = c(\gamma) f(u, U), \quad (u, U) \in F, \tag{21}
$$

for any (real) two-dimensional cycle $\gamma \in F_{u,U}$ where $c(\gamma)$ depends only from the homology class of γ and is not zero for the generic case. It is simple to compute $c(\gamma)$. Let us observe that formula (21) for $V = \text{const}$ is the inversion formula for the Radon transform on $\mathbb{C}P^2$.

Remarks. 1. The cases $\mathcal{A}(\alpha, \beta)$ for other α, β can be handled similarly, but the formulas are more complicated. We will obtain in this way all unitary representations of non-holomorphic discrete series for $SU(2,1)$. The most important point is that instead of the equation $\Box g = 0$ we will have an associated system of first order equations (zero rest mass equations for suitable spinor fields).

2. Our construction is related to the homogeneous (with respect to $SU(2,1)$) domain $\mathcal{D} \subset F$. There are two other homogeneous domains in F: flags (z, Z) with $z \in D_+$ or with $Z \in (D')_+$. These domains fiber over $D_+, (D')_+$ correspondingly, in both cases the fibers are projective lines. They correspond to the holomorphic and antiholomorphic series representations. In the formulas for Cauchy-Szegö kernels (cf. (7)) we will have powers of $\langle w, Z \rangle$ only or $\langle z, W \rangle$ only. Hence, the integral formula will yield holomorphic functions in D_+ or the Penrose transform 1-dimensional cohomology in D_- (holomorphic functions in $(D')_+$).

References

1. A.W. Knapp, *A Szegö Kernel for discrete Series*, In: Proceedings of the International Congress of Mathematics, 1974, 99-104, Vancouver.
2. A. W. Knapp and N. R. Wallach, *Szegö kernels asssociated with discrete series*, Inventiones Math. **34** (1976), 163-300; **62** (1980), 341-346.
3. S. Gindikin, *Holomorphic language for $\bar{\partial}$-cohomology and representations of real semisimple Lie groups*, Contemporary Math. **154** (1993), 103-115.

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Invariant Theory for Boundary Components

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ABSTRACT. This paper, the first in a series devoted to vector-valued harmonic analysis on the boundary of a symmetric cone, concerns finite-dimensional representation theory associated to the boundary components of the cone of positive-definite $n \times n$ Hermitian matrices. The invariant-theoretic results in this paper are crucial to developing a theory of operator-valued Bessel functions on boundary components, which in turn leads to new realizations of singular unitarizable highest weight representations and to connections with the oscillator representation.

1. Introduction

This paper begins a study of vector-valued analysis on the boundary components of a symmetric cone. Here, we develop the finite-dimensional representation theory and invariant theory associated to the boundary components of the cone Ω in $\mathbb{C}^{n\times n}$ of all positive-definite Hermitian matrices. The results in this paper, especially the detailed study of the singular operators on the boundary (Section 5) and the unitary equivalence of Hilbert spaces on the boundary (Section 6) are of independent interest. But this paper is ancillary in nature, preparing the way for later work that will relate operator-valued Bessel functions for boundary components to oscillator representations and to new analytic realizations of the most singular unitarizable highest weight representations for the automorphism groups of Hermitian symmetric spaces. These latter representations may be said to lie in the vector-valued Wallach set (cf., [6]) for these groups.

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The paper is structured as follows. Section 2 treats the boundary components $\partial\Omega_k$ of Ω, $k = 1, 2, \ldots, n-1$. Section 3 is devoted to the finite-dimensional representation theory of the group $A = \mathrm{GL}(n, \mathbb{C})$ that acts on the cone. The main construct is that of a bitriangular structure (N^-, D, N^+) for A attached to a boundary component $\partial\Omega_k$, by means of which we realize certain irreducible finite-dimensional holomorphic representations of A that are "admissible" for $\partial\Omega_k$. These admissible representations π have the property that we can define and analyze the operators $\pi(x)^{-1}$ for $x \in \partial\Omega_k$, despite the fact that $\det x = 0$.

The main theorems of the paper appear in Sections 4 through 6. Section 4 treats the required invariant theory. In particular, we construct all of the N^--invariants (the vectors of "lowest D-type") and relate the spaces of invariants to the boundary components $\partial\Omega_k$. In Section 5 we prove a number of theorems characterizing the image and kernel of the operators $\pi(x)^{-1}$ for $x \in \partial\Omega_k$. Finally, in Section 6 we apply the theorems from earlier sections to establish an important unitary equivalence (Theorem 6.8) for certain Hilbert spaces of vector-valued functions on $\partial\Omega_k$.

Some personal remarks that place the current work in historical perspective are especially appropriate to a volume that honors Ray Kunze. This paper marks the return after a hiatus of nearly a decade, to a central theme of my long collaboration with Ray; namely, the interplay of invariant theory, special functions, and representations of classical groups. In fact, this paper may be seen as a sequel to the first paper we wrote [3] nearly a quarter century ago on operator-valued Bessel functions. In [3] we constructed new unitarizable highest weight representations by analytically continuing the holomorphic discrete series. The thrust of the current work, in this paper and others to follow, is to combine invariant-theoretic constructs along the lines of [4] with the analytic techniques in [3] to realize the remaining unitarizable highest weight representations.

There are two other points of contact of this paper with the literature that should be mentioned. The special case of Theorem 6.8 in which $n = 2$ and $k = 1$ was first proved by H. Jacobsen and M. Vergne [5]. More recently, J. L. Clerc [1] studied vector-valued singular highest weight representations from an analytic point of view. His paper sparked the interest in taking a fresh look at old work.

My closing words are for Ray – my teacher, collaborator, and friend. Suffice it to say that our mathematical association in particular, and our close personal friendship in general, have been a deep source of satisfaction and joy, and constitute one of the high points in my professional life.

2. Boundary Components

2.1 The cone Ω. Let $\mathbb{C}^{n\times n}$ denote the algebra of all complex $n \times n$ matrices, and $J = J_n$ the real subspace of all Hermitian matrices $x = x^*$, where $x^* = \bar{x}^t$. Write $x \geq 0$ to mean that the matrix x is non-negative, and $x > 0$ when x is strictly positive-definite. Then $\Omega = \Omega(n) = \{x \in J : x > 0\}$ is the cone of all positive-definite $n \times n$ matrices over the complex field. The group $A = \mathrm{GL}(n, \mathbb{C})$

of all non-singular $n \times n$ complex matrices acts on Ω by

$$a \cdot x = axa^* \tag{1}$$

and this action is transitive. The unitary subgroup $L = \mathrm{U}(n) = \{l \in A : l^{-1} = l^*\}$ is the subgroup of A fixing the identity matrix $1 = 1_n \in \Omega$, and the map $aL \mapsto x = aa^* = a \cdot 1$ identifies the symmetric space A/L with the cone Ω. In short, $\Omega \cong \mathrm{GL}(n, \mathbb{C})/\mathrm{U}(n)$.

2.2 The boundary components $\partial\Omega_k$. The closure of Ω is $\bar{\Omega} = \{x \in J : x \geq 0\}$ and the boundary $\partial\Omega$ of Ω is $\partial\Omega = \bar{\Omega} - \Omega = \{x \in J : x \geq 0 \text{ and } \det x = 0\}$. The boundary decomposes according to rank as $\partial\Omega = \cup_{k=0}^{n-1} \partial\Omega_k$ where

$$\partial\Omega_k = \{x \in \partial\Omega : \operatorname{rank} x = k\} \tag{1}$$

is the rank k *boundary component*, $k = 0, 1, \ldots, n-1$. Of course, $\partial\Omega_0 = 0$, and our concern in this paper is with the components for $k = 1, 2, \cdots, n-1$. If we block an $n \times n$ matrix in the scheme $(k, n-k)$, then the matrices in $\partial\Omega_k$ have the form

$$x = \begin{pmatrix} x_{11} & x_{21}^* \\ x_{21} & x_{22} \end{pmatrix} \tag{2}$$

in which $x_{11} \in \overline{\Omega(k)}, x_{22} \in \overline{\Omega(n-k)}$, and $x_{21} \in \mathbb{C}^{(n-k)\times k}$. Especially important for our purposes is the dense open subset $\partial\Omega_k^\circ$ of $\partial\Omega_k$ defined by

$$\partial\Omega_k^\circ = \{x \in \partial\Omega_k : \det x_{11} \neq 0\} \tag{3}$$

The group A, acting by (2.1.1), preserves $\partial\Omega_k$; and there exists a relatively A-invariant measure $dm(x)$, unique up to positive multiples, such that

$$dm(a \cdot x) = |\det a|^{2k} dm(x) \tag{4}$$

for all $a \in A$. It can be shown that

$$dm(x) = \frac{dx_{11} dx_{21}}{(\det x_{11})^{n-k}} \tag{5}$$

is such a measure, where dx_{11} and dx_{21} denote Lebesgue measure on $\overline{\Omega(k)}$ and $\mathbb{C}^{(n-k)\times k}$, respectively.

2.3 The orbit space $\mathrm{U}(k)\backslash\mathbb{C}^{k\times n}$. Fix an integer k with $0 < k < n$, and let

$$W = W_k = \{w \in \mathbb{C}^{k\times n} : \operatorname{rank} w = k\}. \tag{1}$$

The subset W is dense and open in $\mathbb{C}^{k\times n}$. We will often write an element of $\mathbb{C}^{k\times n}$ as

$$w = (w_1, w_2) \tag{2}$$

with $w_1 \in \mathbb{C}^{k\times k}$ and $w_2 \in \mathbb{C}^{k\times(n-k)}$. Let $U = \mathrm{U}(k)$ be the $k \times k$ unitary group, acting on $\mathbb{C}^{k\times n}$ by left matrix multiplication. We consider the orbit space

$U\backslash\mathbb{C}^{k\times n}$ for this action, and note that $U\backslash W$ is dense and open in $U\backslash\mathbb{C}^{k\times n}$. The mapping

$$w \mapsto x = w^*w \tag{3}$$

identifies the orbit space $U\backslash W$ with $\partial\Omega_k$, and the action (2.1.1) of $a \in A$ on $\partial\Omega_k$ corresponds to right multiplication on $\mathbb{C}^{k\times n}$ by a^*. That is,

$$wa^* \mapsto a \cdot x \tag{4}$$

Under this identification, Lebesgue measure on $\mathbb{C}^{k\times n}$ corresponds to the measure $dm(x)$ on $\partial\Omega_k$. That is, there exists a positive constant c such that

$$\int_{\partial\Omega_k} f(x)dm(x) = c\int_{\mathbb{C}^{k\times n}} f(w^*w)dw \tag{5}$$

for all integrable functions f on $\partial\Omega_k$. Finally, set

$$W^\circ = \{w \in W : \det w_1 \neq 0\} \tag{6}$$

Under the mapping (3), W° corresponds to $\partial\Omega_k^\circ$. In W-coordinates, the elements (2.2.2) in $\partial\Omega_k$ are given by

$$x = w^*w = \begin{pmatrix} w_1^*w_1 & w_1^*w_2 \\ w_2^*w_1 & w_2^*w_2 \end{pmatrix} \tag{7}$$

Although it is not necessary in our work to have the exact value of the constant c in (5), it can be computed by evaluating the integral on the Gaussian function. The resulting value is

$$c = \frac{2^k(1!)(3!)\cdots(2k-1)!}{\pi^{k(k+1)/2}} \tag{8}$$

2.4 The embedding of $\mathbb{C}^{k\times n}$ in $\mathbb{C}^{n\times n}$. For $w \in W$ given by (2.3.2) we define the $n \times n$ matrix

$$w^\# = \begin{pmatrix} w_1 & w_2 \\ 0 & 0 \end{pmatrix} \tag{1}$$

This embedding of W in $\mathbb{C}^{n\times n}$ commutes with right multiplication; i.e., in reference to (2.3.4)

$$(wa^*)^\# = w^\#a^* \tag{2}$$

for $w \in W$ and $a \in A$. Each $x \in \partial\Omega_k$ decomposes as a product

$$x = w^*w = (w^*)^\#w^\# \tag{3}$$

in which the U-orbit of w is uniquely determined by x, and the mapping $x \mapsto w^\#$ identifies x with $w^\#$. Under this identification, $a \cdot x \mapsto w^\#a^*$. Note that the matrices $w^\#$ in which $\det w_1 \neq 0$ correspond to the subset $\partial\Omega_k^\circ$ given by (2.2.3).

2.5 The subgroup B. Take

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1}$$

as a fixed point in $\partial\Omega_k$, $1 = 1_k$ being the $k \times k$ identity matrix. To each $w = (w_1, w_2) \in W^\circ$ we associate the matrix

$$b = b(w) = \begin{pmatrix} w_1^* & 0 \\ w_2^* & 1 \end{pmatrix}. \tag{2}$$

Then, as w varies through W°, the matrix

$$x = b \cdot e = w^* w = (w^*)^\# (w^\#) \tag{3}$$

varies through $\partial\Omega_k^\circ$. Thus, the subgroup

$$B = \{b = b(w) : w \in W^\circ\} \tag{4}$$

of A acts transitively on $\partial\Omega_k^\circ$. Let T be the group of all upper triangular $k \times k$ matrices

$$t = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1k} \\ 0 & t_{22} & \cdots & t_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{kk} \end{pmatrix} \tag{5}$$

in which $t_{jj} > 0$ for all $j = 1, \cdots, k$; and let B_0 be the subgroup of B consisting of all matrices of the form

$$b_0 = b(t, s) = \begin{pmatrix} t^* & 0 \\ s^* & 1 \end{pmatrix} \tag{6}$$

with $t \in T$ and $s \in \mathbb{C}^{k \times (n-k)}$. Since each $w_1 \in \mathrm{GL}(k, \mathbb{C})$ can be written uniquely as $w_1 = ut$ with $u \in U = \mathrm{U}(k)$ and $t \in T$, it follows that B_0 acts simply transitively on $\partial\Omega_k^\circ$. Put another way, the matrices $(t, s) \in \mathbb{C}^{k \times n}$ such that $t \in T$ are a complete set of representations for the orbit space $U \backslash W^\circ$. In the coordinates t, s of B_0, the measure (2.2.5) has the form

$$dm(x) = 2^k t_{11}^{2k-1} t_{22}^{2k-3} \cdots t_{k-1,k-1}^3 t_{kk}\, dt\, ds \tag{7}$$

where $x = b_0 \cdot e$ and dt and ds are Lebesgue measures on T and $\mathbb{C}^{k \times (n-k)}$, respectively.

As a final remark, we observe that

$$w^\# = eb^* \text{ and } (w^\#)^* = be \tag{8}$$

where $w^\#$ is given by (2.4.1) and b by (2).

3. $\partial\Omega_k$-admissible representations

3.1 Algebraic induction. Let A be a complex algebraic group. A *bitriangular structure* for A is a triple (N^-, D, N^+) of complex algebraic subgroups having the following three properties:
(a) $N^\pm$ are unipotent subgroups, each of which is normalized by D.
(b) N^-D and N^+D are semi-direct products of D with N^- and N^+, respectively.
(c) The multiplication $(n^-, d, n^+) \mapsto a = n^-dn^+$ is a biregular mapping from $N^- \times D \times N^+$ to a dense open subset of A.

The method of algebraic induction [4; §6] provides an explicit construction of all the irreducible finite-dimensional holomorphic representations π of A in terms of certain irreducible finite-dimensional representations λ of D that we refer to as *inductive* (with respect to the bitriangular structure). In particular, the mapping $\lambda \mapsto \pi = \pi_\lambda$ parametrizes the finite-dimensional algebraic dual of A.

We are interested specifically in the group $A = \mathrm{GL}(n, \mathbb{C})$. A function on A is *regular* if it lies in the algebra generated by the entry functions a_{ij} together with the function $1/\det a$; and a regular function is *polynomial* if it does not involve $1/\det a$. A representation of A is *algebraic* if the matrix entries are regular functions. We use the abbreviation *ifdh* for an irreducible finite-dimensional holomorphic representation of A. Such a representation is necessarily algebraic, so the mapping $\lambda \mapsto \pi = \pi_\lambda$ actually parametrizes the finite-dimensional holomorphic dual of A.

In its most familiar setting the bitriangular structure is "minimal"; that is, D is the diagonal (so-called Cartan) subgroup of A, and $N^\pm D$ are the opposed upper and lower triangular (Borel) subgroups of A. Here, algebraic induction is a variant of Cartan's theorem of the highest (or lowest) weight; the irreducible representations λ are all one-dimensional and of the form $\lambda = \lambda_m$ where

$$\lambda_m(d) = d_1^{m_1} d_2^{m_2} \cdots d_n^{m_n} \tag{1}$$

for $d = \mathrm{diag}(d_1, \cdots, d_n) \in D$; and λ is inductive if and only if

$$m_1 \geq \cdots \geq m_n. \tag{2}$$

The representation $\pi = \pi_m$ of A corresponding to λ_m is said to have *signature* $m = (m_1, \cdots, m_n)$, and λ_m is usually known as the *lowest weight character* of π. In particular, the n-tuples m satisfying (2) parametrize the ifdh representations of $A = \mathrm{GL}(n, \mathbb{C})$.

For our purposes in this paper, however, we require $n-1$ different bitriangular structures, each being maximal rather than minimal. Each of these bitriangular structures is associated to a boundary component $\partial\Omega_k$ of Ω, $k = 1, 2, \cdots, n-1$.

3.2 The bitriangular structure for $\partial\Omega_k$. Fix a boundary component $\partial\Omega_k$, $0 < k < n$, and write an $n \times n$ matrix according to the $(k, n-k)$ blocking scheme

as

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{1}$$

We associate to $\partial\Omega_k$ the bitriangular structure (N^-, D, N^+) in which N^- and N^+ are the opposed unipotent subgroups of matrices

$$n^-(\xi) = \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix} \quad \text{and} \quad n^+(\eta) = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix} \tag{2}$$

respectively, where $\xi \in \mathbb{C}^{(n-k)\times k}$ and $\eta \in \mathbb{C}^{k\times(n-k)}$; and D is the diagonal block subgroup of matrices

$$d = d(\alpha, \beta) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \tag{3}$$

with $\alpha \in \mathrm{GL}(k, \mathbb{C})$ and $\beta \in \mathrm{GL}(n-k, \mathbb{C})$. For $a \in A$ with $\det a_{11} \neq 0$,

$$a = n^-(a_{21}a_{11}^{-1})d(a_{11}, a_{22} - a_{21}a_{11}^{-1}a_{12})n^+(a_{11}^{-1}a_{12}) \tag{4}$$

and the mapping $(n^-, d, n^+) \mapsto a = n^-dn^+$ is seen to be biregular from $N^- \times D \times N^+$ to the dense open subset $\{a \in A : \det a_{11} \neq 0\}$ of A.

3.3 The induced representations π_λ. We now describe the induction procedure for the bitriangular structure in paragraph 3.2. We refer to [4; §6] for details and proofs in the general setting. Let λ be an ifdh representation of D on the Hilbert space $\mathcal{V} = \mathcal{V}_\lambda$, and let $\mathcal{L} = \mathcal{L}_\lambda$ be the space of all linear transformations on $\mathcal{V}$. We assume that the inner product on $\mathcal{V}$ is chosen such that

$$\lambda(d^*) = \lambda(d)^* \tag{1}$$

for all $d \in D$. In light of (3.4.4), we set

$$\lambda^\#(a) = \lambda(d(a_{11}, a_{22} - a_{21}a_{11}^{-1}a_{12}))^{-1} \tag{2}$$

for all $a \in N^-DN^+$, and say that λ is *inductive* if $\lambda^\#$ extends from N^-DN^+ to a regular function $\lambda^\# : A \to \mathcal{L}$.

Denote by $\mathcal{P}(A, \lambda)$ the space of all regular functions $f : A \to \mathcal{V}$ satisfying the covariance property

$$f(adn^+) = \lambda(d)^{-1}f(a) \tag{3}$$

for all $(a, d, n^+) \in A \times D \times N^+$. From the density of N^-DN^+ in A, if λ is inductive then $\mathcal{P}(A, \lambda) \neq 0$, and the formula

$$(\pi(a_\circ)f)(a) = f(a_\circ^{-1}a) \tag{4}$$

for $a_\circ \in A$ and $f \in \mathcal{P}(A, \lambda)$ defines an ifdh representation $\pi = \pi_\lambda$ of A. We say that π is the *representation of A algebraically induced from λ*. We set $\mathcal{H}_\pi =$

$\mathcal{P}(A,\lambda)$, and we write $\mathcal{H} = \mathcal{H}_\pi$ when there will be no confusion. We introduce on $\mathcal{H}$ the inner product

$$\langle f_1|f_2\rangle = \int_A \langle f_1(a)|f_2(a)\rangle e^{-\mathrm{tr}(aa^*)}da \tag{5}$$

for $f_1, f_2 \in \mathcal{H}$, where da is Lebesgue measure. Clearly the restriction of π to the unitary subgroup $L = \mathrm{U}(n)$ is unitary, and by analytic continuation

$$\pi(a^*) = \pi(a)^* \tag{6}$$

for all $a \in A$.

In this paper we do not use the concept of signature; however, for completeness we identify the representations π_λ according to signature (cf., (3.1.1) *et seq.*). Write

$$\lambda(d(\alpha,\beta)) = \lambda_1(\alpha) \otimes \lambda_2(\beta) \tag{7}$$

where λ_1 and λ_2 are ifdh representations of $\mathrm{GL}(k,\mathbb{C})$ and $\mathrm{GL}(n-k,\mathbb{C})$, respectively, and let the signature of λ_1 be $(m_1,\dots,m_k)$ and that of λ_2 be $(m_{k+1},\dots,m_n)$. Then λ is inductive if and only if $m_k \geq m_{k+1}$, and the signature of π_λ is $(m_1,\cdots,m_n)$.

3.4 $\partial\Omega_k$-admissible representations. From (3.2.3) we see that the group D splits as the direct product $D = D_1D_2$ with $D_1 = \{d_1 = d_1(\alpha) = d(\alpha,1) : \alpha \in \mathrm{GL}(k,\mathbb{C})\}$ and $D_2 = \{d_2 = d_2(\beta) = d(1,\beta) : \beta \in \mathrm{GL}(n-k,\mathbb{C})\}$. We can identify D_1 with $\mathrm{GL}(k,\mathbb{C})$ and D_2 with $\mathrm{GL}(n-k,\mathbb{C})$, and write λ according to (3.3.7) as the tensor product of a representation λ_1 of $\mathrm{GL}(k,\mathbb{C})$ and a representation λ_2 of $\mathrm{GL}(n-k,\mathbb{C})$.

An ifdh representation λ of D will be called *$\partial\Omega_k$-admissible* if two properties hold:

(a) The restriction of λ to D_2 is the identity.

(b) The mapping $\alpha \mapsto \lambda(d_1(\alpha))^{-1}$ is polynomial.

Property (a) means that $\lambda_2 = 1$ in (3.3.7). Equivalently,

$$\lambda(d(\alpha,\beta)) = \lambda_1(\alpha) \tag{1}$$

for all $\alpha \in \mathrm{GL}(k,\mathbb{C})$ and $\beta \in \mathrm{GL}(n-k,\mathbb{C})$, and we can write $\mathcal{V}_{\lambda_1}$ for $\mathcal{V}_\lambda$ and $\mathcal{L}_{\lambda_1}$ for $\mathcal{L}_\lambda$. From property (b) we define a polynomial mapping $\mu : \mathbb{C}^{k\times k} \mapsto \mathcal{L}_{\lambda_1}$ such that

$$\mu(\alpha) = \lambda_1(\alpha)^{-1} \tag{2}$$

for $\alpha \in \mathrm{GL}(k,\mathbb{C})$, and

$$\mu(\alpha_1\alpha_2) = \mu(\alpha_2)\mu(\alpha_1) \tag{3}$$

for $\alpha_1, \alpha_2 \in \mathbb{C}^{k\times k}$.

Note when λ is $\partial\Omega_k$-admissible, formula (3.3.2) for $\lambda^\#$ has the simple form

$$\lambda^\#(a) = \mu(a_{11}) \tag{4}$$

for $a \in \mathbb{C}^{n\times n}$; and the defining covariance condition (3.3.3) for $\mathcal{H}_\pi = \mathcal{P}(A,\lambda)$ becomes

$$f(ad(\alpha,\beta)n^+) = \mu(\alpha)f(a) \tag{5}$$

for $a \in A$, $d = d(\alpha,\beta) \in D$, and $n^+ \in N^+$. Moreover, a function $f \in \mathcal{H}_\pi$ extends uniquely from A to a polynomial function, again denoted f, on all of $\mathbb{C}^{n\times n}$; and (5) holds for all $a \in \mathbb{C}^{n\times n}, \alpha \in \mathbb{C}^{k\times k}, \beta \in \mathbb{C}^{(n-k)\times(n-k)}$, and $n^+ \in N^+$.

When λ is $\partial\Omega_k$-admissible, we will also say that the induced representation $\pi = \pi_\lambda$, defined by (3.3.4), is $\partial\Omega_k$-admissible. We mention in passing that if the signature of λ_1 is $(m_1,\dots,m_k)$, then $m_k \geq 0$ and the signature of π_λ is $(m_1,\dots,m_k,0,\dots,0)$.

4. Vectors of lowest D-type

4.1 N^-–invariants. Let $\pi = \pi_\lambda$ be $\partial\Omega_k$-admissible and set $\mathcal{H} = \mathcal{H}_\pi$. We define λ_1 by (3.4.1) and μ by (3.4.2), and we denote the space of N^-–invariants by

$$\mathcal{H}^- = \{f \in \mathcal{H} : \pi(n^-)f = f \text{ for all } n^- \in N^-\}. \tag{1}$$

From (3.2.4), (3.4.5), and the density of N^-DN^+ in A, if $f \in \mathcal{H}^-$ then

$$f(a) = \mu(a_{11})f(1) \tag{2}$$

for all $a \in A$. This motivates the following characterization of $\mathcal{H}^-$.

4.2 Theorem. *(a) Let $v \in \mathcal{V} = \mathcal{V}_{\lambda_1}$, and define $f_v : A \to \mathcal{V}$ by*

$$f_v(a) = \mu(a_{11})v. \tag{1}$$

Then $f_v \in \mathcal{H}^-$.
(b) The mapping $\Phi : v \mapsto f_v$ is an isomorphism of $\mathcal{V}$ with $\mathcal{H}^-$.
(c) For any $v \in \mathcal{V}$,

$$\pi(n^-d(\alpha,\beta))f_v = f_{\lambda(\alpha)v} \tag{2}$$

for all $n^- \in N^-$ and $d = d(\alpha,\beta) \in D$.
(d) There exists a positive constant c depending only upon n, k, and λ_1, such that

$$\langle f_{v_1} | f_{v_2}\rangle = c\langle v_1 | v_2\rangle \tag{3}$$

for all $v_1, v_2 \in \mathcal{V}$.

PROOF. From (3.2.4), the density of N^-DN^+ in A, and (3.4.3), it follows that f_v satisfies the covariance property (3.4.5). Since f_v is regular, we have shown that $f_v \in \mathcal{H}$. Since $(n^-a)_{11} = a_{11}$ for all $n^- \in N^-$ and $a \in A$, it follows that $f_v \in \mathcal{H}^-$. This proves part (a). Part (b) is a consequence of (4.1.2). Part (c) follows from the identity $((n^-d(\alpha,\beta))^{-1}a)_{11} = \alpha^{-1}a_{11}$. We prove (3). From (3.3.5) and (1),

$$\langle f_{v_1} | f_{v_2}\rangle = \langle M_{\lambda_1} v_1 | v_2\rangle$$

where

$$M_{\lambda_1} = \int_A e^{-\mathrm{tr}(aa^*)} \mu(a_{11}a_{11}^*) da.$$

We integrate in the variables a_{12}, a_{21}, and a_{22} to obtain

$$M_{\lambda_1} = c_1 \int_{\mathrm{GL}(k,\mathbb{C})} e^{-\mathrm{tr}(a_{11}a_{11}^*)} \mu(a_{11}a_{11}^*) da_{11}$$

with $c_1 = \pi^{n^2-k^2}$. Introduce polar coordinates $a_{11} = r^{1/2}u$ with $u \in U = \mathrm{U}(k)$ and $r = a_{11}a_{11}^* \in \Omega(k)$. From [3;(4.8)], there exists a constant c_2 such that $da_{11} = c_2 du dr$ with du normalized Haar measure on U and dr Lebesgue measure on $\Omega(k)$. Then

$$M_{\lambda_1} = c_1 c_2 \Gamma_{\lambda_1}(k) \tag{4}$$

where

$$\Gamma_{\lambda_1}(k) = \int_{\Omega(k)} e^{-\mathrm{tr}\ r} \lambda_1(r)^{-1} dr$$

is the gamma function for the cone $\Omega(k)$. This integral converges absolutely to a scalar matrix [2;§2.4], which establishes (3) and completes the proof.

We remark that although our work does not require the exact value of the constant c in (3), that constant can be readily calculated. In fact, the constants appearing in (4) have the following values:

$$c_1 = \pi^{n^2-k^2} \quad \text{and} \quad c_2 = 2^k \pi^{k(k+1)/2} \prod_{j=1}^{k-1} j!$$

and

$$\Gamma_{\lambda_1}(k) = 2^{-k} \pi^{k(k-1)/2} \prod_{j=1}^{k} (m_j + k - j)!,$$

where λ_1 has signature $(m_1, \cdots, m_k)$. Hence,

$$c = \pi^{n^2} \prod_{j=1}^{k} \frac{(m_j + k - j)!}{(k-j)!}. \tag{5}$$

We make two remarks related to parts (c) and (d) of the theorem. First, formula (2) can be rephrased as the intertwining property

$$\Phi \pi(n^- d(\alpha, \beta)) \Phi^{-1} = \lambda_1(\alpha). \tag{6}$$

Secondly, formula (3) is equivalent to the statement that the map $v \mapsto c^{-1/2} f_v$ is unitary from $\mathcal{V}$ to $\mathcal{H}^-$.

4.3 Definition. In light of (4.2.2) – and in analogy to the classical terminology used in the context of the minimal bitriangular structure (cf., paragraph 3.1) – we refer to elements of $\mathcal{H}^-$ as *vectors of lowest D-type.*

4.4 The complementary space $\mathcal{H}^{(+)}$. Let $\mathcal{H}^{(+)}$ be the subspace of $\mathcal{H}$ spanned by the set $\{\pi(n^+)f - f : f \in \mathcal{H}, n^+ \in N^+\}$. If $f^- \in \mathcal{H}^-$, then

$$\langle f^-|\pi(n^+)f - f\rangle = \langle \pi(n^+)^* f^-|f\rangle - \langle f^-|f\rangle = \langle f^-|f\rangle - \langle f^-|f\rangle = 0$$

since $(n^+)^* \in N^-$. Thus, $\mathcal{H}^{(+)} \perp \mathcal{H}^-$. That $\mathcal{H}$ is spanned by $\mathcal{H}^-$ and $\mathcal{H}^{(+)}$ is proved in [4; Theorem 6.6]. We have, then, the following important result.

4.5 Theorem. *$\mathcal{H}$ decomposes as the orthogonal direct sum*

$$\mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}^{(+)}. \tag{1}$$

4.6 The projection on $\mathcal{H}^-$. Since μ is polynomial, from (4.2.1) and (3.3.4) we conclude that the mapping $a \to \pi(a)^{-1}$ is polynomial. Thus, there exists a polynomial map $\nu = \nu_{\lambda_1}$ from $\mathbb{C}^{n\times n}$ to linear transformations on $\mathcal{H}$ such that

$$\nu(a) = \pi(a)^{-1} \tag{1}$$

for $a \in A$, and

$$\nu(a_1 a_2) = \nu(a_2)\nu(a_1) \tag{2}$$

for $a_1, a_2 \in \mathbb{C}^{n\times n}$. Recall the fixed point $e \in \Omega$ given by (2.5.1), and set

$$E = \nu(e). \tag{3}$$

4.7 Theorem. *E is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}^-$.*

PROOF. We need to show that $Ef_v = f_v$ for all $v \in \mathcal{V}$, and $Ef = 0$ for all $f \in \mathcal{H}^{(+)}$. We rewrite (4.2.2) in the form

$$\nu(n^- d(\alpha,\beta))f_v = f_{\mu(\alpha)v} \tag{1}$$

for $n^- \in N^-$ and $d(\alpha,\beta) \in D$. Since both sides are polynomial in α and β, the identity (1) holds even when α and β are singular. Since $e = d(1,0)$, we see from (1) that $Ef_v = \nu(d(1,0))f_v = f_{\mu(1)v} = f_v$. Next, we use the identity $n^+e = e$ for all $n^+ \in N^+$, to see that $E = \nu(e) = \nu(n^+e) = \nu(e)\nu(n^+) = E\nu(n^+)$. Thus, $E\pi(n^+) = E$, or equivalently $E(\pi(n^+)f - f) = 0$ for all $f \in \mathcal{H}$, from which it follows that $Ef = 0$ for all $f \in \mathcal{H}^{(+)}$.

5. The operators $\nu(x)$ for $x \in \partial\Omega_k$

5.1 The operators $\nu(x)$ for singular x. Let $\pi = \pi_{\lambda_1}$ be a $\partial\Omega_k$-admissible representation of A, acting in the space $\mathcal{H} = \mathcal{H}_\pi$, and let $\nu = \nu_{\lambda_1}$ be the polynomial mapping on $\mathbb{C}^{n\times n}$ given by (4.6.1). Since in general π is not polynomial, $\pi(x)$ is not defined if $\det x = 0$. However, $\nu(x)$ is always defined, even when $\det x = 0$. We are interested, in particular, in these operators when $x \in \partial\Omega_k$, and we wish to calculate the kernel and image of $\nu(x)$ for $x \in \partial\Omega_k$.

Recall from paragraph 2.4. that each $x \in \partial\Omega_k$ can be written as $x = (w^\#)^* w^\#$ for some $w \in W = \{w = (w_1, w_2) \in \mathbb{C}^{k\times n} : \text{rank } w = k\}$. Of course, $\det w^\# = 0$, so it is natural to first describe the kernel and image of the operators $\nu(w^\#)$ and

$\nu((w^{\#})^*)$. We make two simple but important preliminary observations. First, from (2.4.1) and (2.5.1), $ew^{\#} = w^{\#}$ for all $w \in \mathbb{C}^{k\times n}$; hence

$$\nu(w^{\#})E = \nu(w^{\#}) \tag{1}$$

where E is the orthogonal projection of $\mathcal{H}$ onto the space $\mathcal{H}^-$ of vectors of lowest D-type. Secondly, for any operator T on a Hilbert space

$$\text{kernel}\, T = (\text{image}\, T^*)^{\perp}. \tag{2}$$

Finally, we recall from (2.3.6) that W° consists of those $w = (w_1, w_2) \in W$ for which $\det w_1 \neq 0$.

5.2 Theorem. *(a) For any $w \in W^\circ$*

$$\text{image}\, \nu((w^{\#})^*) = \mathcal{H}^-. \tag{1}$$

Moreover, the restriction of $\nu((w^{\#})^*)$ to $\mathcal{H}^-$ is an isomorphism of $\mathcal{H}^-$.
(b) For any $w \in W^\circ$

$$\text{kernel}\, \nu(w^{\#}) = \mathcal{H}^{(+)} \tag{2}$$

and the restriction of $\nu(w^{\#})$ to $\mathcal{H}^-$ is injective.
(c) For any $w \in W^\circ$,

$$\text{image}\, \nu(w^{\#}) = \nu(w^{\#})\mathcal{H}^- \tag{3}$$

and

$$\text{kernel}\, \nu((w^{\#})^*) = (\nu(w^{\#})\mathcal{H}^-)^{\perp}. \tag{4}$$

PROOF. Part (b) follows by (5.1.2) from (1); equation (3) is immediate from (2); and (4) follows from (3) by (5.1.2). Thus, we are left with the proof of part (a). Let $v \in \mathcal{V}_{\lambda_1}$. We first observe from (4.7.1) that

$$\nu((w^{\#})^*)f_v = f_{\lambda_1(w_1^{\vee})v} \tag{5}$$

where $w_1^{\vee}$ is the contragredient matrix $w_1^{\vee} = (w_1^*)^{-1}$. Thus $\nu((w^{\#})^*)$ maps $\mathcal{H}^-$ isomorphically onto itself. In particular, we know that $\mathcal{H}^- \subset \text{image}\, \nu((w^{\#})^*)$, and all that remains is to prove the reverse containment. Taking adjoints on both sides of (5.1.1), we see that $E\nu((w^{\#})^*) = \nu((w^{\#})^*)$, which proves that $\text{image}(\nu((w^{\#})^*)) \subset \mathcal{H}^-$.

5.3 Theorem. *Let $x \in \partial\Omega_k^\circ$ and write x according to (2.5.3) as $x = b \cdot e$ with $b \in B$. Then*

$$\text{kernel}\, \nu(x) = \pi(b)\mathcal{H}^{(+)} \tag{1}$$

and

$$\text{image}\, \nu(x) = \pi(b^{\vee})\mathcal{H}^- \tag{2}$$

where $b^\vee = (b^*)^{-1}$.

PROOF. Since $\nu(x)^* = \nu(x)$, (2) follows via (5.1.2) from (1). We are left with proving (1). Let $f \in \mathcal{H}$. Then $f \in \operatorname{kernel} \nu(x)$ if and only if $\langle \nu(x)f|f'\rangle = 0$ for all $f' \in \mathcal{H}$. Since

$$\langle \nu(x)f|f'\rangle = \langle v(b\cdot e)f|f'\rangle = \langle \nu(beb^*)f|f'\rangle = \langle \nu(b^*)E\nu(b)f|f'\rangle = \langle \nu(b)f|E\nu(b)f'\rangle$$

we see that $f \in \operatorname{kernel} \nu(x)$ if and only if $\nu(b)f \perp \mathcal{H}^-$; or equivalently, $\nu(b)f \in \mathcal{H}^{(+)}$. Since $\nu(b) = \pi(b^{-1})$, $f \in \operatorname{kernel} \nu(x)$ if and only if $f \in \pi(b)\mathcal{H}^{(+)}$.

If we combine Theorems 5.2 and 5.3 we obtain the following results.

5.4 Corollary. *Let* $x \in \partial\Omega^\circ$ *and write* x *as* $x = w^*w$ *with* $w \in W^\circ$. *Then*

$$\operatorname{image} \nu(x) = \operatorname{image} \nu(w^\#) = \nu(w^\#)\mathcal{H}^- \tag{1}$$

and

$$\operatorname{kernel} \nu(x) = \operatorname{kernel} \nu((w^\#)^*) = (\nu(w^\#)\mathcal{H}^-)^\perp. \tag{2}$$

PROOF. From (5.3.1), (5.2.2), and (2.5.8), $\operatorname{image} \nu(x) = \nu(b^*)\mathcal{H}^- = \nu(b^*)E\mathcal{H}^- = \nu(eb^*)\mathcal{H}^- = \nu(w^\#)\mathcal{H}^- = \operatorname{image} \nu(w^\#)$. This proves (1), and (2) follows from (5.1.2).

5.5 Corollary. *Let* $x \in \partial\Omega_k^\circ$, *and view* $E\nu(x)E$ *as a linear transformation on* $\mathcal{H}^-$. *Then* $E\nu(x)E$ *is positive-definite.*

PROOF. Write $x = w^*w$ with $w \in W^\circ$, and let $v \in \mathcal{V}_{\lambda_1}$ with $v \neq 0$. Then by part (a) of Theorem 5.2, $\langle E\nu(x)Ef_v|f_v\rangle = \langle \nu(x)f_v|f_v\rangle = ||\nu((w^\#)^*)f_v||^2 > 0$.

6. The Hilbert space $L^2_\nu(\partial\Omega_k)$

6.1 The pre-Hilbert space $\mathbf{L^2}_\nu(\partial\Omega_k)_{\mathrm{pre}}$. We apply the results of Section 5 to infinite-dimensional analysis on $\partial\Omega_k$. Let $\pi = \pi_{\lambda_1}$, as usual, be a $\partial\Omega_k$-admissible representation of A, and define ν by (4.6.1). Let $dm(x)$ be the relatively invariant measure (2.2.5) on $\partial\Omega_k$. We denote by $L^2_\nu(\partial\Omega_k)_{\mathrm{pre}}$ the space of all measurable functions $\phi : \partial\Omega_k \to \mathcal{H}_\pi$ such that

$$\int_{\partial\Omega_k} \langle v(x)\phi(x)|\phi(x)\rangle dm(x) < \infty. \tag{1}$$

The formula

$$\langle \phi_1|\phi_2\rangle_\nu = \int_{\partial\Omega_k} \langle \nu(x)\phi_1(x)|\phi_2(x)\rangle dm(x) \tag{2}$$

defines a non-negative sesquilinear form on $L^2_\nu(\partial\Omega_k)_{\mathrm{pre}}$ which fails to be an inner product because there exist non-zero vectors of length 0. That is, there exist null vectors. Let

$$\mathcal{N}_\nu = \{\phi \in L^2_\nu(\partial\Omega_k)_{\mathrm{pre}} : ||\phi||_\nu = 0\} \tag{3}$$

be the subspace of null vectors, and

$$L^2_\nu(\partial\Omega_k)^\sim = L^2_\nu(\partial\Omega_k)_{\text{pre}}/\mathcal{N}_\nu \tag{4}$$

the quotient space. Then $L^2_\nu(\partial\Omega_k)^\sim$ is a Hilbert space. We want to give an explicit concrete realization of this Hilbert space.

6.2 Lemma. *A function $\phi \in L^2_\nu(\partial\Omega_k)_{\text{pre}}$ is a null vector if and only if $\phi(x) \in$ kernel $\nu(x)$ for a.e. $x \in \partial\Omega_k$.*

PROOF. If $\phi(x) \in \text{kernel}\,\nu(x)$ a.e., then from (6.1.1) it follows that $\phi \in \mathcal{N}_\nu$. Conversely, suppose $\phi \in \mathcal{N}_\nu$ and write $x \in \partial\Omega_k^\circ$ as $x = w^*w$ with $w \in W^\circ$. Then

$$||\phi||^2_\nu = \int_{\partial\Omega_k} \langle v(x)\phi(x)|\phi(x)\rangle dm(x) = \int_{\partial\Omega_k} ||\nu((w^\#)^*)\phi(x)||^2 dm(x) = 0$$

and we see that $\phi(x) \in \text{kernel}\,\nu((w^\#)^*)$ for a.e. x. The lemma now follows from (5.4.2).

6.3. Lemma. *For each $x \in \partial\Omega_k$, let $\mathcal{H}^1(x) =$ image $\nu(x)$ and let $P_1(x)$ be the orthogonal projection of $\mathcal{H} = \mathcal{H}_\pi$ onto $\mathcal{H}^1(x)$. For any $\phi \in L^2_\nu(\partial\Omega_k)_{\text{pre}}$ define the function ϕ_1 by $\phi_1(x) = P_1(x)(\phi(x))$. Then $\phi_1 \in L^2_\nu(\partial\Omega_k)_{\text{pre}}$ and $||\phi - \phi_1|| = 0$; i.e., $\phi = \phi_1$ modulo $\mathcal{N}_\nu$.*

PROOF. Set $\mathcal{H}^2(x) = \text{kernel}\,\nu(x)$ and let $P_2(x)$ be the projection of $\mathcal{H}$ onto $\mathcal{H}^2(x)$. For $\phi \in L^2_\nu(\partial\Omega_k)_{\text{pre}}$, set $\phi_2(x) = P_2(x)(\phi(x))$. Since $\mathcal{H} = \mathcal{H}^1(x) \oplus \mathcal{H}^2(x)$ for each $x \in \partial\Omega_k$, it follows that $\phi(x) = \phi_1(x) + \phi_2(x)$. By Lemma 6.2, $\phi_2 \in \mathcal{N}_\nu$ and $\phi = \phi_1$ modulo $\mathcal{N}_\nu$.

6.4. Definition. Let $L^2_\nu(\partial\Omega_k)$ be the space of all measurable functions $\phi : \partial\Omega_k \to \mathcal{H}_\pi$ such that

$$\phi(x) \in \text{image}\,\nu(x) \tag{1}$$

for a.e. x, and such that (6.1.1) holds.

6.5 Theorem. *Equipped with the inner product (6.1.2), $L^2_\nu(\partial\Omega_k)$ is a Hilbert space that realizes $L^2_\nu(\partial\Omega_k)^\sim$. That is, the mapping $\phi \mapsto \phi + \mathcal{N}_\nu$ is unitary from $L^2_\nu(\partial\Omega_k)$ to $L^2_\nu(\partial\Omega_k)^\sim$.*

That $L^2_\nu(\partial\Omega_k)$ is a Hilbert space follows from the usual Riesz-Fisher type argument. That $L^2_\nu(\partial\Omega_k)$ is unitarily equivalent to $L^2_\nu(\partial\Omega_k)^\sim$ is merely a rephrasing of Lemma 6.3.

6.6 A unitary equivalence. Let $x \in \partial\Omega_k^\circ$, and write $x = w^*w$ with $w \in W^\circ$. If $\phi \in L^2_\nu(\partial\Omega_k)$, then by (5.4.1) there exists a measurable function $h : \mathbb{C}^{k\times n} \to \mathcal{V}_{\lambda_1}$ such that

$$\phi(x) = \nu(w^\#) f_{h(w)} \tag{1}$$

for a.e. x. Next, let

$$\tau(w) = ww^* = w_1w_1^* + w_2w_2^* \tag{2}$$

for $w = (w_1, w_2) \in W^\circ$. Evidently, $\tau(w)$ is a $k \times k$ matrix that lies in the cone $\Omega(k)$, so $\tau(w)$ is invertible. That is, $\tau(w) \in \mathrm{GL}(k, \mathbb{C})$. From (2.4.1),

$$w^\#(w^\#)^* = \begin{pmatrix} \tau(w) & 0 \\ 0 & 0 \end{pmatrix}. \tag{3}$$

Then from (1), (2.4.1), and (4.2.3),

$$\begin{aligned} \langle \nu(x)\phi(x) | \phi(x) \rangle &= ||\nu((w^\#)^*)\phi(x)||^2 = ||\nu((w^\#)^*)\nu(w^\#) f_{h(w)}||^2 \\ &= ||\nu(w^\#(w^\#)^*) f_{h(w)}||^2 = \left|\left|\nu\left(\begin{pmatrix} \tau(w) & 0 \\ 0 & 0 \end{pmatrix}\right) f_{h(w)}\right|\right|^2 \\ &= ||f_{\lambda_1(\tau(w))^{-1}h(w)}||^2 = c||\lambda_1(\tau(w))^{-1}h(w)||^2 \end{aligned}$$

where c is the constant (4.2.5). Thus, by (2.3.5)

$$\int_{\partial\Omega_k} \langle \nu(x)\phi(x) | \phi(x) \rangle dm(x) = C \int_{\mathbb{C}^{k\times n}} ||\lambda_1(\tau(w))^{-1}h(w)||^2 dw \tag{4}$$

for all $\phi \in L^2_\nu(\partial\Omega_k)$, where C is the product of the constants (4.2.5) and (2.3.8).

We are led, therefore, to define the following Hilbert space.

6.7 Definition. Let $L^2(\mathbb{C}^{k\times n}, \lambda_1)$ be the space of all measurable functions $h : \mathbb{C}^{k\times n} \to \mathcal{V}_{\lambda_1}$ such that for a.e. $w \in \mathbb{C}^{k\times n}$

$$h(uw) = \lambda_1(u)h(w) \tag{1}$$

for all $u \in U = \mathrm{U}(k)$, and such that

$$\int_{\mathbb{C}^{k\times n}} ||h(w)||^2 \, dw < \infty.$$

Equipped with the inner product

$$\langle h_1 | h_2 \rangle = \int_{\mathbb{C}^{k\times n}} \langle h_1(w) | h_2(w) \rangle \, dw \tag{2}$$

$L^2(\mathbb{C}^{k\times n}, \lambda_1)$ is a Hilbert space.

The following result is a consequence of (6.6.1) and (6.6.4).

6.8 Theorem. *For $h \in L^2(\mathbb{C}^{k\times n}, \lambda_1)$ define $\psi : \partial\Omega_k^\circ \to \mathcal{H}_\pi$ by*

$$\psi(x) = C^{-1/2}\nu(w^\#) f_{\lambda_1(\tau(w))h(w)} \tag{1}$$

where $x = w^*w$ with $w \in W^\circ$, and C is the product of the constants (4.2.5) and (2.3.8). Then the mapping $h \mapsto \psi$ is a unitary equivalence of $L^2(\mathbb{C}^{k\times n}, \lambda_1)$ with $L^2_\nu(\partial\Omega_k)$.

The unitary equivalence in Theorem 6.8 is the foundation for a variety of results in harmonic analysis and infinite-dimensional representation theory related to the boundary components of Ω. For example, Theorem 6.8 underlies the relationship among operator-valued Bessel functions on the boundary components $\partial\Omega_k$, singular highest weight representations of the group $\mathrm{U}(n, n)$, and the oscillator representation acting on $L^2(\mathbb{C}^{k\times n})$ for $0 < k < n$. These ideas, and

their extension to symmetric cones more generally, will be the subject of future papers.

REFERENCES

1. J.L. Clerc, *Reproducing kernels for some highest weight representations*, preprint, 1993.
2. K.I. Gross and H. Ding, *Operator-valued Bessel functions on Jordan algebras*, J. reine angew. Math. **435** (1993), 157-196.
3. K.I. Gross and R. A. Kunze, *Fourier-Bessel transforms and holomorphic discrete series*, Springer Lecture Notes, #266 (1971), 79-122.
4. ———, *Finite-dimensional induction and new results on invariants for classical groups*, Amer. J. Math. **106** (1984), 893-974.
5. H.P. Jacobsen and M. Vergne, *Wave and Dirac Operators, and representations of the conformal group*, J. Funct. Anal. **24** (1977), 52-106.
6. N. Wallach, *The analytic continuation of the holomorphic discrete series, I and II*, Trans. Amer. Math. Soc. **251** (1979), 1-17 & 18-37.

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Decomposition of Exterior Algebras

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Dedicated to Ray A. Kunze

ABSTRACT. Consider the group $GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$ acting on $\mathbb{C}^p \otimes \mathbb{C}^q$. We ask: How does $\Lambda^*(\mathbb{C}^p \otimes \mathbb{C}^q)$ decompose under the induced action of $GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$? In this paper, we determine the structure of the $GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$-module $\Lambda^*(\mathbb{C}^p \otimes \mathbb{C}^q)$.

1. Introduction

Consider the group $GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$ acting on $\mathbb{C}^p \otimes \mathbb{C}^q$. We ask: How does $\Lambda^*(\mathbb{C}^p \otimes \mathbb{C}^q)$ decompose under the induced action of $GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$? Whereas the decomposition of $P(\mathbb{C}^p \otimes \mathbb{C}^q)$, the polynomial algebra on $\mathbb{C}^p \otimes \mathbb{C}^q$, has been analyzed by many authors, little has been said about the decomposition of $\Lambda^*(\mathbb{C}^p \otimes \mathbb{C}^q)$. In this paper, we determine the structure of the $GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$-module $\Lambda^*(\mathbb{C}^p \otimes \mathbb{C}^q)$. It is interesting to note that Lie algebra cohomology plays the same role in the decomposition of $\Lambda^*(\mathbb{C}^p \otimes \mathbb{C}^q)$ that Capelli identities play in the decomposition of $P(\mathbb{C}^p \otimes \mathbb{C}^q)$. In fact, there is a variant of Bott's "strange equality" (equation 15.3 of [B]) proved by Kostant [K] that underlies the spirit of our program. We first change our perspective to consider a more general situation.

Let G be a compact connected Lie group with K a closed connected subgroup so that G/K is a Hermitian symmetric space with respect to the action of G. If T is a maximal toroidal subgroup of K, T is also a maximal toroidal subgroup of G. We may assume that T, K, and G are the respective real forms of complex groups $T_{\mathbb{C}}$, $K_{\mathbb{C}}$, and $G_{\mathbb{C}}$. If $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{k}$ is the Lie algebra of K, we have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}, \quad \mathfrak{g}_c = \mathfrak{k}_c + \mathfrak{p}_c$$

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where

$$\mathfrak{p}_c = \mathfrak{p}_+ + \mathfrak{p}_-,\ [\mathfrak{p}_\pm, \mathfrak{p}_\pm] = 0,\ [\mathfrak{p}_\mp, \mathfrak{p}_\pm] \subseteq \mathfrak{k}_c \text{ and } [\mathfrak{k}_c, \mathfrak{p}_\pm] \subseteq \mathfrak{p}_\pm\ .$$

Let $W(K)$ denote the Weyl group of K and $W(G)$ the Weyl group of G, let N be a maximal nilpotent subgroup of $K_{\mathbb{C}}$ normalized by $T_{\mathbb{C}}$ and let $H^*(G/K, \mathbb{C})$ be the deRham cohomology of G/K. Then our form of the strange equality is

$$\dim H^*(G/K, \mathbb{C}) = |W(G)|/|W(K)| = \dim(\Lambda^* \mathfrak{p}_\pm)^N\ .$$

The first equality has been established by several authors in several different ways and we will not elaborate on it. In [K], Kostant explains Bott's strange equality and establishes the second equality. One of our goals will be to give another proof of this second equality. We now elaborate.

Consider the space

$$I_r = \mathrm{Hom}_K(\Lambda^r \mathfrak{p}_c, \mathbb{C}) = (\Lambda^r \mathfrak{p}_c)^K\ .$$

Since there is a $Z \in \mathfrak{k}_c$ such that $\mathrm{ad}Z|\mathfrak{p}_\pm = \pm 1$, we see that $I_r = 0$ if r is odd, while

$$I_{2r} = \mathrm{Hom}_K(\Lambda^{2r} \mathfrak{p}_c, \mathbb{C}) \cong \mathrm{End}_K(\Lambda^r \mathfrak{p}_+)\ .$$

Hence $H^{2r}(G/K, \mathbb{C}) = I_{2r}$, and $H^{2r+1}(G/K, \mathbb{C}) = 0$. Let τ_r be the set of irreducible holomorphic representations of $K_{\mathbb{C}}$ occurring in $\Lambda^r \mathfrak{p}_+$. As a $K_{\mathbb{C}}$-module

$$\Lambda^r \mathfrak{p}_+ = \sum_{D \in \tau_r} m(D) V_D\ ,$$

where V_D is the irreducible module corresponding to D occurring with multiplicity $m(D)$ in $\Lambda^r \mathfrak{p}_+$. Setting $m = \dim \mathfrak{p}_\pm$, $h = \dim H^*(G/K, \mathbb{C})$, and $p = \dim(\Lambda^* \mathfrak{p}_+)^N$, we have $h = \sum_{r=0}^{m} \sum_{D \in \tau_r} m(D)^2$ and $p = \sum_{r=0}^{m} \sum_{D \in \tau_r} m(D)$. We see that $h = p$, if and only if $m(D) = 1$ for all D and all r. (This is in fact Kostant's result.) We will give a new proof of this result.

Our method here is to appeal to the trivial inequality $p \leq h$, and thus use h as an *a priori* cap on the possible dimension of $(\Lambda^* \mathfrak{p}_+)^N$. We will in fact prove $p = h$ by explicitly constructing $|W(G)|/|W(K)|$ linearly independent elements of $(\Lambda^* \mathfrak{p}_+)^N$. This will determine the complete $K_{\mathbb{C}}$-module structure of $(\Lambda^* \mathfrak{p}_+)$. Our technique involves a case by case study of each irreducible Hermitian symmetric space of compact type and is reminiscent of Weyl's use of Capelli's identities.

Finally I must add that these results have been obtained independently by Carey Rader [R].

2. The classical cases

Recall the six irreducible Hermitian symmetric spaces of compact type:

(a) $U(p+q)/U(p) \times U(q)$,
(b) $Sp(n)/U(n)$,

(c) $SO(2n)/U(n)$,
(d) $SO(n+2)/SO(n) \times SO(2)$,
(e) $E_6/\text{Spin}\,(10) \cdot T$, and
(f) $E_7/E_6 \cdot T$.

We now examine each case.

(a) $G = U(p+q)$, $K = U(p) \times U(q)$, $|W(G)| = (p+q)!$, $|W(K)| = p!q!$
The map of $U(p) \times U(q)$ to K given by mapping

$$(A, B) \to (A, {}^tB^{-1})$$

is an isomorphism. The representation of $GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$ on $\mathfrak{p}_+$ is the standard action on $\mathbb{C}^p \otimes \mathbb{C}^q$. Let $\{e_1, \cdots, e_p\}$ be the standard basis of $\mathbb{C}^p$, and let $\{f_1, \cdots, f_q\}$ denote the standard basis of $\mathbb{C}^q$. Let

$$Y(p, q) = \{m \in Z^p : q \geq m_1 \geq \cdots \geq m_p \geq 0\} .$$

Consider a $p \times q$ board of squares where (i, j) stands for the square in the i^{th}-row and the j^{th}-column. For $m \in Y(p, q)$, we say $(i, j) \in m$ if $j \leq m_i$. For $m \in Y(p, q)$ set

$$v(m) = \bigwedge_{(i,j) \in m} (e_i \otimes f_j) .$$

Let $\mathfrak{n}_1$ (resp. $\mathfrak{n}_2$) be the maximal nilpotent subalgebra of $\mathfrak{gl}(p, \mathbb{C})$ (resp. $\mathfrak{gl}(q, \mathbb{C})$) consisting of strictly upper triangular matrices. Then $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ is a maximal nilpotent subalgebra of $\mathfrak{gl}(p, \mathbb{C}) \oplus \mathfrak{gl}(q, \mathbb{C})$. Now $\mathfrak{n}_1$ (resp. $\mathfrak{n}_2$) is spanned by the $p \times p$-matrices (resp. $q \times q$-matrices) E_{kl}^* (resp. F_{kl}) $(k < l)$ whose only nonzero entry is a 1 in the k^{th}-row and l^{th}-column. Since $(k, j) \in m$ whenever $(l, j) \in m$ we have $E_{kl}v(m) = 0$ for $E_{kl} \in \mathfrak{n}_1$, and since $(i, k) \in m$ whenever $(i, l) \in m$ we have $F_{kl}v(m) = 0$ for $F_{kl} \in \mathfrak{n}_2$. Thus $v(m)$ is the highest weight vector of an irreducible $GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$-module in $\Lambda^{|m|}\mathfrak{p}_+$. Note that $m \in Y(p, q)$ is the Young diagram of an irreducible representation of $GL(p, \mathbb{C})$, and the transposed diagram m^t, formed by making the i^{th}-column of m the i^{th}-row of m^t, represents a Young diagram of $GL(q, \mathbb{C})$. The irreducible module with highest weight vector $v(m)$ is just $m \otimes m^t$. If $H_1 = \text{diag}\,(x_1, \cdots, x_p) \in \mathfrak{gl}(p, \mathbb{C})$ and $H_2 = \text{diag}\,(y_1, \cdots y_q) \in \mathfrak{gl}(q, \mathbb{C})$, we have

$$(H_1 + H_2)v(m) = \left[\sum_{i-1}^{p} m_i x_i + \sum_{j=1}^{q} m_j^t y_j\right] v(m).$$

Hence, if m, m' are in $Y(p, q)$ and $m \neq m'$, the corresponding representations are inequivalent. To show that $Y(p, q)$ gives a complete parametrization of the $GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$-module structure of $\Lambda^*\mathfrak{p}_+$ it suffices to show that $|Y(p, q)| = (p+q)!/p!q!$. This is easily accomplished by fixing p and using induction on q.
This answers the question posed in the introduction.

(b) $G = Sp(n)$, $K = U(n)$, $|W(G)|/|W(K)| = 2^n$.

The action of $GL(n,\mathbb{C})$ on $\mathfrak{p}_+$ is the action of $GL(n,\mathbb{C})$ on $S^2(\mathbb{C}^n)$. Let $Y(Sp(n),\ U(n)) = \{m \in Z^n : n \geq m_1 \geq \cdots \geq m_n \geq 0$ where $m_j > m_{j+1}$ whenever $m_j > 0\}$. For each m, we say $(i,j) \in m$ if $m_i > 0$ and $i \leq j \leq i+m_i-1$. For a given m, set

$$v(m) = \bigwedge_{(i,j)\in m} e_i \cdot e_j \qquad (e_i \cdot e_j = e_i \otimes e_j + e_j \otimes e_i)\ .$$

As in (a), $v(m)$ is the highest weight vector of an irreducible representation of $GL(n,\mathbb{C})$, and if $m \neq m'$, the corresponding representations are inequivalent. A simple calculation yields $|Y(Sp(n),\ U(n))| = 2^n$. Hence we have the complete decomposition of $\Lambda^*\mathfrak{p}_+$.

(c) $G = SO(2n)$, $K = U(n)$, $|W(G)|/|W(K)| = 2^{n-1}$.

As a $GL(n,\mathbb{C})$-module, $\mathfrak{p}_+$ is isomorphic to $\Lambda^2\mathbb{C}^n$. Let $Y(Sp(2n),\ U(n)) = \{m \in Z^{n-1} : n-1 \geq m_1 \geq \cdots \geq m_{n-1} \geq 0$ where $m_j > m_{j+1}$ whenever $m_j > 0\}$. For $m \in Y(SO(2n),\ U(n))$, say $(i,j) \in m$ if $i < j \leq m_i + i$ and set

$$v(m) = \bigwedge_{(i,j)\in m} (e_i \Lambda e_j) \qquad (e_i \Lambda e_j = e_i \otimes e_j - e_j \otimes e_i)\ .$$

As in (a), $v(m)$ is the highest weight vector of an irreducible $GL(n,\mathbb{C})$-module, and if $m \neq m'$, the corresponding representations are inequivalent. Since $|Y(SO(2n),\ U(n))| = 2^{n-1}$, we have the decomposition of $\Lambda^*\mathfrak{p}_+$.

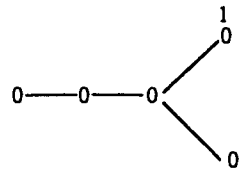
(d) $G = SO(n+2)$, $K = SO(n) \times SO(2)$, $|W(G)|/|W(K)| = n+2$ when n is even and $n+1$ when n is odd. In this case the representation of $K_{\mathbb{C}}$ on $\mathfrak{p}_+$ is just the standard representation of $SO(n,\mathbb{C})$ on $\mathbb{C}^n$ times a scalar. The representation of $K_{\mathbb{C}}$ on $\Lambda^r\mathfrak{p}_+$ is the standard representation of $SO(n,\mathbb{C})$ on $\Lambda^r\mathbb{C}^n$ times a scalar and so is always irreducible except for the case where $2r = n$, and in this case the representation splits into the direct sum of two irreducible representations (see [J] §6). This completes our picture of the classical types.

3. The exceptional types

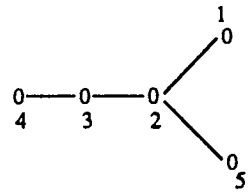
Until now the weight graph of the representation of $K_{\mathbb{C}}$ on $\mathfrak{p}_+$ has not played an obvious role in our picture. It is virtually impossible to discuss the exceptional types without reference to the weight graph. Although we did not refer to the weight graph explicitly in making our calculations for the classical types, it in reality made our calculations possible.

(e) $G = E_6$, $K = \mathrm{Spin}(10) \times T$, $|W(G)|/|W(K)| = 27$.

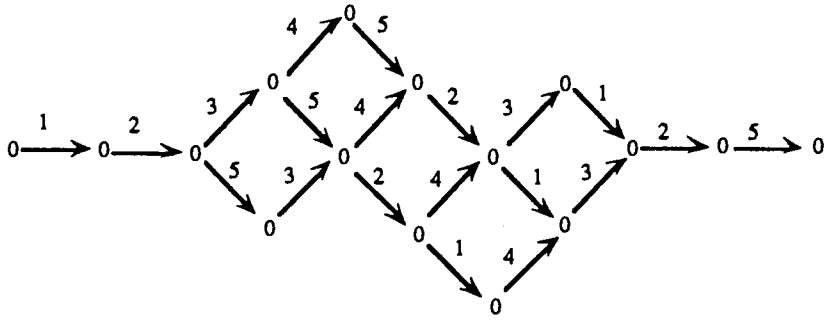
The representation of Spin(10) on $\mathfrak{p}_+$ is a half spin representation



The root diagram of Spin(10) is



and the weight graph of the representation of Spin(10) on $\mathfrak{p}_+$ is



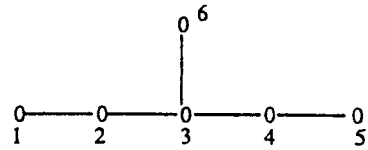
The initial dot represents the highest weight λ. The second dot is obtained by subtracting α_1 from λ, etc. A subgraph is called *acceptable* if whenever a dot q is contained in the subgraph, the subgraph contains any dot p if an arrow starts at p and ends at q. Clearly, any acceptable subgraph corresponds to the highest weight vector of an irreducible representation of Spin(10) on $\Lambda^*\mathfrak{p}_+$. The degree of an acceptable subgraph will denote the number of dots in the subgraph, and an acceptable subgraph of degree r corresponds to an irreducible representation of Spin(10) in $\Lambda^r\mathfrak{p}_+$. Rather than list all possible acceptable subgraphs, the interested reader may verify the following chart, where the top number indicates the degree and the bottom number indicates the number of acceptable subgraphs of that degree, determines all acceptable subgraphs.

$0 \leq \deg \leq 3$ or $13 \leq \deg \leq 16$	$4 \leq \deg \leq 7$ or $9 \leq \deg \leq 12$	$\deg = 8$
1	2	3

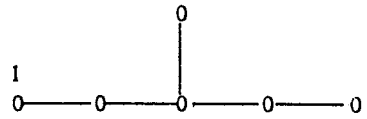
Each dot of the weight graph represents an element of $\mathfrak{p}_+$. The initial dot represents the highest weight vector of $\mathfrak{p}_+$. If $0 \xrightarrow{i} 0$ occurs in the graph with the first dot representing v and the second dot representing w, then w is a nonzero multiple of $e_{-i}v$ where e_{-i} is the ith negative simple root vector. Thus, if X is an acceptable subgraph consisting of r dots representing $v_1, \ldots, v_r$, the vector $v_1 \wedge \cdots \wedge v_r$ is a highest weight vector of $\Lambda^r\mathfrak{p}_+$.

(f) $G = E_7$, $K = E_6 \cdot T$, $|W(G)|/|W(K)| = 56$.

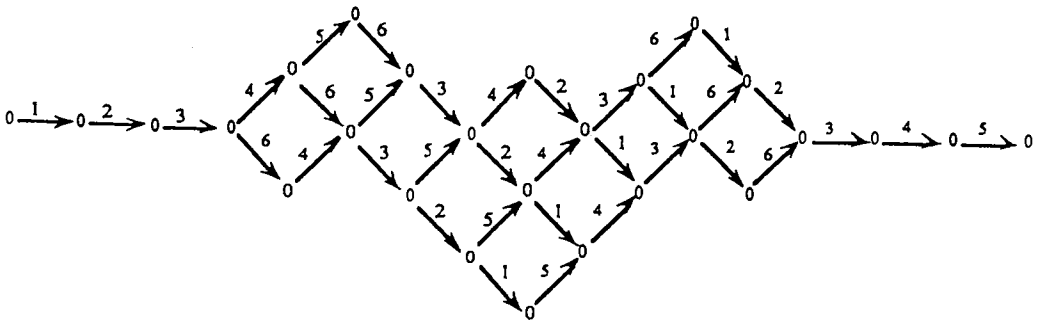
The Dynkin diagram of E_6 is



The representation of E_6 on $\mathfrak{p}_+$ is



and the corresponding weight graph is



Again the interested reader may verify the following chart:

$0 \leq \deg \leq 4$ or $23 \leq \deg \leq 27$	$5 \leq \deg \leq 8$ or $19 \leq \deg \leq 22$	$9 \leq \deg \leq 18$
1	2	3

4. Concluding Remarks

Although the weight graph was not mentioned explicitly in the classical cases, it is clear that we used the weight graph extensively in all cases. It is also clear that the weight graph is used to obtain a first approximation to $(\Lambda^* V)^N$ for

a K-module V, since any acceptable subgraph determines a unique element of $(\Lambda^* V)^N$. It is somewhat fortuitous that this first order approximation succeeds in accomplishing our complete task in the Hermitian symmetric space case.

REFERENCES

[B] R. Bott, *Homogeneous vector bundles*, Annals of Math. **66** (1957), 203-248.

[H] S. Helgason, *Differential Geometry and Symmetric Spaces*, Academic Press, New York, 1962.

[J] K. Johnson, *A constructive approach to tensor product decompositions*, J. reine angew. Math. **388** (1988), 129-148.

[K] B. Kostant, *Lie algebra cohomology and generalized Schubert cells*, Annals of Math. **77** (1963), 72-144.

[R] C. Rader, *The Betti numbers of classical symmetric spaces*, (preprint).

[T] M. Takeuchi, *Cell decomposition and Morse equalities on certain symmetric spaces*, J. Fac. Sci. Univ. Tokyo, Sect. I, **10** (1964), 88-123.

[W] H. Weyl, *The Classical Groups*, Princeton University Press, Princeton, 1946.

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On a Mean Value Property for Hyperbolic Spaces

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To Ray Kunze on his sixty-fifth birthday

ABSTRACT. On $\mathbb{R}^n$, or on a Riemannian symmetric space, let h be a positive radial function of total integral one. If f is harmonic and $f * h$ is meaningful, then clearly $f * h = f$. The problem under discussion is whether the converse statement holds.

1. Introduction

In [1] Ahern, Flores and Rudin prove the following surprising result. Let B_n be the unit ball in $\mathbb{C}^n$ with the Bergman metric, i.e. the standard ball model of Hermitian hyperbolic space. Clearly, if f is harmonic on B_n with respect to the intrinsic metric, then so is $f \circ g$ for every g in the isometry group G, and it follows that

$$f(g(0)) = \int_{B_n} (f \circ g) d\nu \quad (g \in G) \tag{1.1}$$

where ν is Lebesgue measure normalized so that $\nu(B_n) = 1$. The question asked in [1] is whether the property (1.1) characterizes the harmonic functions belonging to $L^1(\nu)$, and the answer found is *yes* when $n \leq 11$, and *no* when $n \geq 12$.

In the present article this result is generalized to all (i.e. real, complex, quaternionic and octonionic) hyperbolic spaces and even beyond, to a class of harmonic spaces, with the aid of a partly new unified proof. There is also a brief discussion of an analogous question in Euclidean space.

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Key words and phrases. Harmonic functions, hyperbolic spaces, mean value property, Riemannian symmetric spaces of rank one.

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Our main goal in looking at this problem was to try to understand the reason for the curious dependence on the dimension in the result of [1]. Without claiming to have been completely successful in this, we can make the following remarks.

From an intrinsic geometric viewpoint the measure ν is rather arbitrary; it can be written as hdv where dv is the G-invariant measure and h is a certain positive radial function with $\int hdv = 1$. The property (1.1) then takes the form

$$f * h = f \tag{1.2}$$

where the star denotes convolution with respect to dv; lifting the functions to G this is the usual group-convolution.

Now the problem can be posed quite in general, and makes sense for $\mathbb{R}^n$ as well as for the hyperbolic spaces: Given a fixed radial function $h \geq 0$ with $\int hdv = 1$, do the conditions $f \circ g \in L^1(hdv), (g \in G)$ and (1.2) together imply that f is harmonic? A by now classical result of Furstenberg [7] says that if we require $f \in L^\infty$ then the answer is *yes* for *any* fixed h on any symmetric space. The situation is, however, quite different for $L^p(hdv)$ with $1 \leq p < \infty$.

The solutions of (1.2) form a linear space invariant under translations by G. Since h is not assumed to have compact support, the known results about "spectral synthesis" and "spectral analysis" do not apply, but it is still reasonable to start looking for solutions that are eigenfunctions of the intrinsic Laplacian Δ. (As it is proved *a posteriori* in [1], and more generally here, in the case of the hyperbolic spaces these span the space of all solutions.) For such f the problem immediately reduces to finding a $\lambda \in \mathbb{C}$, not corresponding to the eigenvalue 0, such that $\hat{h}(\lambda) = 1$ (with $\hat{h}$ denoting the spherical Fourier transform of h) and such that there exist eigenfunctions corresponding to λ in $L^1(hdv)$. When h is very small at infinity, every eigenfunction is in $L^1(hdv)$ and $\hat{h}$ is an entire function; since $\hat{h}$ does assume the value 1 (at the λ corresponding to eigenvalue 0), it assumes it infinitely many times. It is in the cases where h is just large enough so that $L^1(hdv)$ contains some but not all unbounded eigenfunctions that the situation becomes delicate and the dimension of the space begins to play a decisive role.

It is a pleasure for the author to acknowledge his debt to S. Helgason for some useful discussions and for some references to the literature.

2. A Euclidean Analogue

This section is perhaps not too interesting for its own sake; it is included here because it shows in a simple case some of the phenomena that occur in the hyperbolic spaces.

Denoting $|x| = (\sum x_j^2)^{1/2}$ for $x \in \mathbb{R}^n$, we fix the function

$$h(x) = c_n e^{-2\pi|x|} \tag{2.1}$$

where $c_n = \pi^{(n+1)/2}/\Gamma(\frac{n+1}{2})$ is chosen so that $\int h(x)dx = 1$. We note that, for fixed y, $h(x+y)/h(x)$ is bounded in x; hence, if $f \in L^1(h(x)dx)$ then also $f \circ g \in L^1(h(x)dx)$ for all g in the Euclidean motion group G. Clearly, the property $\int (f \circ g)h = f(g(0)), (g \in G)$ is equivalent to the equation $f * h = f$.

We denote by Δ the Euclidean Laplacian,

$$\Delta = \sum \frac{\partial^2}{\partial x_j^2}.$$

PROPOSITION 2.1. *Let h be as in (2.1). The equation $f * h = f$ has solutions $f \neq 0$ satisfying $f \in L^1(h(x)dx)$ and*

$$\Delta f = -4\pi^2 \lambda^2 f \tag{2.2}$$

with some $\lambda \neq 0$ if and only if $n \geq 9$.

PROOF. If such a solution f exists, then translating it to be non-zero at 0 and averaging it over the rotation group K we may assume that it is radial. As is well-known, the only radial solution of (2.2) is (up to a constant factor)

$$j_\lambda(x) = \int_k e^{2\pi i \lambda (e_1 | k \cdot x)} dx$$

where $e_1 = (1, 0, \ldots, 0)$. From the asymptotic behavior of Bessel functions [4, vol. II, 7.13 (3)] it follows that, for large $|x|$,

$$j_\lambda(x) \sim c|x|^{-(n+1)/2} e^{2\pi |Im\, \lambda||x|}$$

From this it is clear that $j_\lambda \in L^p(h(x)dx)$ if and only if $|Im\, \lambda| < \frac{1}{p}$. In fact, this can also be seen by first proving, by elementary arguments, that j_λ is in L^p if and only if $e^{2\pi |Im\, \lambda|(e_1|x)}$ is, and then using the Schwarz inequality.

By the general version of the Godement mean value theorem [9, p. 402] every solution f of (2.2) satisfies

$$\int_k f(x + ky)dk = j_\lambda(y) f(x)$$

for all $x, y \in \mathbb{R}^n$. Multiplying by $h(y)(= h(-y))$ and integrating with respect to y we obtain

$$f * h = \hat{h}(\lambda e_1) f$$

where $\hat{h}$ is the Fourier transform of h. Explicitly,

$$\hat{h}(\lambda e_1) = (1 + \lambda^2)^{-(n+1)/2}$$

with that branch of the square root which is positive on the positive half-line.

In view of these remarks the existence of an f as in the statement of the proposition is equivalent to the existence of a $\lambda \neq 0$ with $|Im\, \lambda| < 1$ and such that $(1 + \lambda^2)^{-\frac{n+1}{2}} = 1$.

The range of $1 + \lambda^2$ for $|Im\, \lambda| < 1$ is the interior of the parabola $Re\, z > (\frac{1}{2} Im\, z)^2$, and the question is whether this contains any point $z \neq 1$ such that $z^{(n+1)/2} = 1$. It is clear that if such a z exists at all, then $e^{4\pi i/(n+1)}$ is such a

z. Now this point is inside our parabola if and only if $n \geq 9$. This finishes the proof.

REMARKS. If in Proposition 2.1 we replace L^1 by L^p for some $1 < p < \infty$, then λ is further restricted, and n has to be larger for non-trivial solutions to exist. It is clear that for any fixed $1 \leq p < \infty$ the number of λs giving solutions tends to infinity with n. It is easy and instructive to see what happens if we change h to $h(x) = e^{-\pi|x|^2}$. This h is much too small: For every $\lambda \in \mathbb{C}, j_\lambda$ is in every $L^p(h(x)dx)$ if $p < \infty$, and the Fourier transform $\hat{h}(\lambda e_1) = e^{-\pi\lambda^2}$ is equal to 1 for infinitely many values of λ, independently of the dimension.

We do not pursue the Euclidean case any further; in particular we leave open the question whether Proposition 2.1 remains true without the condition (2.2).

3. The Hyperbolic Spaces

The hyperbolic spaces are exactly the Riemannian symmetric spaces of non-compact type having rank one. As proved by Takeuchi [13], they all have canonical realizations as a unit ball B_m in some $\mathbb{R}^m$. They are explicitly described in [11]; another way to construct them is indicated in [2].

We fix a symmetric space X of rank one and, using the standard notations (cf. e.g. [9]) we write G for the connected isometry group, o for a chosen base point, K for the stabilizer of o , $\{a_t\}$ for a one-parameter group of transvections based at o. Then $\{a_t o\}$ is a geodesic line and every point in the space appears in the form $ka_t \cdot o \quad (k \in K, t \in \mathbb{R})$. The space has two important invariants, p and q, the multiplicities of the short resp. long restricted roots. (Actually, real hyperbolic space fits into the picture in two ways: with either $p = 0$ or $q = 0$. We will always consider it in the version $p = 0$; this gives in our later setup the Poincaré model, while $q = 0$ would give the less popular Cayley-Klein model.) The real dimension of the space is

$$m = p + q + 1,$$

another important quantity is given by

$$\rho = \frac{p}{2} + q.$$

The canonical realization of X is given by the map $ka_t o \mapsto (tanht)k \cdot e_1$ where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^m$ and k acts on $\mathbb{R}^m$ in the same way as on the tangent space of X at o. The intrinsic volume element of X is known to be [5],[10]

$$dv = \delta(t)dtdk = (\sinh t)^{p+q}(\cosh t)^q dtdk \tag{3.1}$$

where dk is normalized Haar measure on K. (We have chosen a simple normalization of dv; the literature is not uniform on this point.)

In the canonical ball realization, with $r = |x| = \tanh t$ this becomes

$$dv = (1 - r^2)^{-(\rho+1)} r^{m-1} drdk = \frac{1}{m}(1 - |x|^2)^{-(\rho+1)} d\nu(x) \tag{3.2}$$

where ν is Lebesgue measure normalized to give $\nu(B_m) = 1$. Hence, in terms of the coordinate t we can write

$$d\nu = m(\cosh t)^{-2(\rho+1)}dv. \tag{3.3}$$

Now, generalizing [1] and using the Euler beta function, we define for $k > -1$ (not necessarily an integer),

$$(T_k f)(g \cdot o) = \frac{2}{mB(\frac{m}{2}, k+1)} \int_{B_m} (1 - |x|^2)^k f(gx) d\nu(x) \tag{3.4}$$

for $f \in L^1(B_m, \nu)$. (When the same B_m is the model for two different symmetric spaces this formula is still the same for both; what is different is the way in which G acts). To rewrite T_k in terms of the intrinsic coordinates we define, for $k > -1$,

$$h_k(ka_t \cdot o) = \frac{2}{B(\frac{m}{2}, k+1)} (\cosh t)^{-2(\rho+1+k)}. \tag{3.5}$$

We have $\int h_k dv = 1$, $d\nu = h_0 dv$, and we can write

$$T_k f = f * h_k \tag{3.6}$$

for $f \in L^1(h_0 dv)$. We note that, since $\frac{1}{2}e^t < \cosh t < e^t$ and since t is the geodesic distance, $h_0(g^{-1}y)/h_0(y)$ is bounded on X for fixed $g \in G$. Therefore the integral in (3.5) (and in (3.6)) is well defined.

We write Δ for the invariant Laplacian of X, and we note that [8],[5],[10] for radial functions written in terms of t the effect of Δ can be computed by applying the ordinary differential operator

$$\frac{d^2}{dt^2} + \{(p+q)\coth t + q\tanh t\}\frac{d}{dt}. \tag{3.8}$$

PROPOSITION 3.1. *The equation $f * h_0 = f$ has non-zero solutions satisfying $f \in L^1(h_0 dv)$ and*

$$\Delta f = \mu f \tag{3.9}$$

with some complex $\mu \neq 0$ if and only if $\rho \geq 12$.

PROOF. Writing $\mu = -(\lambda^2 + \rho^2)$, it is well known that (3.9) has a unique K - invariant solution having value 1 at o, the spherical function $\phi_\lambda (= \phi_{-\lambda})$. It is known ([8], also [5],[10]) that $\phi_\lambda(a_t \cdot o) = F(a, b; c; z)$ where $z = -\sinh^2 t$ and

$$a = \frac{\rho + i\lambda}{2}, \quad b = \frac{\rho - i\lambda}{2}, \quad c = \frac{m}{2}.$$

Since we are working with the ball model where the natural variable is $x = \tanh t$, it may be nicer to rewrite this, using $r^2 = z/(z-1)$ and a Kummer relation, in the form

$$\phi_\lambda(a_t \cdot o) = (1 - r^2)^a F(a, c - b; c; r^2). \tag{3.10}$$

Now the solution space of (3.9) contains elements in $L^1(h_0 dv)$ if and only if $\phi_\lambda \in L^1(h_0 dv)$. By (3.2) this happens if and only if

$$\int_0^1 (1-r^2)^a |F(a, c-b; c; r^2)| r^{m-1} dr < \infty.$$

By [4, Vol. I, p. 76, (9)], for large $z, F(a,b;c;z) \sim \lambda_1 z^{-a} + \lambda_2 z^{-b}$ (with a logarithmic factor if $a-b$ is an integer). Using $r^2 = z/(z-1)$ this gives that our integral is convergent exactly when $Re\, a$, $Re\, b > -1$, i.e. when

$$|Im\, \lambda| < \rho + 2. \tag{3.11}$$

(By similar arguments it is easy to see that $\phi_\lambda \in L^p(h_k dv)$ if and only if $|Im\, \lambda| < \rho + 2(1+k)/p$.)

By the generalized Godement mean value theorem [9, p.402] the solutions of (3.9) satisfy the identity

$$\int_K f(gk \cdot y) dk = \phi_\lambda(y) f(g \cdot o).$$

(We need the following only for $k = 0$, but with the same effort, in order to hint at some generalizations, we do the general case:) If f is also in $L^1(h_k dv)$, we multiply the identity by $h_k(y) (= h_k(y^{-1}))$ and integrate over X to obtain, for $|Im\, \lambda| < \rho + 2(1+k)/p$,

$$(f * h_k)(g \cdot o) = \hat{h}_k(\lambda) f(g \cdot o). \tag{3.12}$$

Here $\hat{h}_k$ is the spherical Fourier transform; using (3.10),(3.2) and the substitution $x = r^2$ we have

$$\begin{aligned} \hat{h}_k(\lambda) &= \int_X h_k(y) \phi_\lambda(y) dv(y) \\ &= \frac{2}{B(\frac{m}{2}, k+1)} \int_0^1 x^{\frac{m}{2}-1} (1-x)^{k+a} F(a, c-b; c; x) dx \end{aligned}$$

To compute the integral we use [4, Vol. I, p. 78, 2.4 (2)]:

$$\int_0^1 x^{s-1} (1-x)^{c'-s-1} F(a, b'; s, xz) dx = \frac{\Gamma(s)\Gamma(c'-s)}{\Gamma(c')} F(a, b'; c'; z)$$

with $s = \frac{m}{2}, b' = c-b, c' = k+a+\frac{m}{2}-1, 0 < z < 1$. (The formula applies, since $Re\, c'$, $Re\, s > 0$ by the condition on λ.) Letting $z \to 1$ and using

$$F(a, b'; c'; 1) = \frac{\Gamma(c')\Gamma(c'-a-b')}{\Gamma(c'-a)\Gamma(c'-b)}$$

(cf. [4, Vol. I, p. 104, (46)]) we finally obtain

$$\hat{h}_k(\lambda) = \frac{\Gamma(k+1+\frac{\rho+i\lambda}{2})\Gamma(k+1+\frac{\rho-i\lambda}{2})}{\Gamma(k+1)\Gamma(k+1+\rho)}. \tag{3.13}$$

(Another way to evaluate $\hat{h}_k(\lambda)$ is sketched in [10, p. 70].)

In view of (3.12) all that is needed to prove our proposition is to decide whether $\hat{h}_0(\lambda)$ assumes the value 1 for some $\lambda \neq \pm i\rho, |Im\,\lambda| < \rho + 2$. Now the function (3.13) (for $k = 0$) is carried into the function $\phi_\rho(\beta)$ of [1] by the substitution $\beta = (\rho + i\lambda)/2$, and [1, Prop. 3.8] gives the answer, *yes* if $\rho \geq 12$ and *no* if $\rho \leq 11$.

REMARK. The proof also shows that the possible μ's in (3.9) always form a discrete set, namely the image under the map $\lambda \mapsto -(\lambda^2 + \rho^2)$ of the zeroes of the function $\hat{h}_0(\lambda) - 1$ contained in the strip $|Im\,\lambda| < \rho + 2$.

THEOREM 3.2. *The equation $f * h_0 = f$ has non-harmonic solutions in $L^1(h_0 dv)$ if and only if $\rho \geq 12$.*

PROOF. Our proof follows [1]; it is only necessary to establish that the main ingredients remain true in our more general situation.

First we must see that the operators $T_k (k > 0)$ are bounded on $L^1(h_0 dv)$. By the inequality

$$\|T_k f\|_1 = \int_G \left| \int_G f(y) h_k(y^{-1}x) dy \right| h_0(x) dx \leq$$
$$\leq \int_G \left(\frac{1}{h_0(y)} \int_G h_k(y^{-1}x) h_0(x) dx \right) |f(y)| h_0(y) dy$$

this will follow if we can prove that

$$h_0 * h_k \leq M h_0 \tag{3.14}$$

with some constant M. (Note that G is unimodular and the functions are even.) For this we use the expression for the convolution of radial functions from [6] (cf. also [10, p. 59]), which gives

$$(h_0 * h_k)(a_t \cdot o)$$
$$= \int_0^\infty \int_0^\infty (\cosh r)^{-2(\rho+1)} (\cosh s)^{-2(\rho+k+1)} K(r,s,t) \delta(r) \delta(s) ds dt \tag{3.15}$$

with $K(r,s,t) = 0$ unless $|r - s| \leq t \leq r + s$, in which case

$$K(r,s,t) = \frac{(\cosh r \cosh s \cosh t)^{\frac{p}{2}-1}}{(\sinh r \sinh s \sinh t)^{p+q-1}} K'(r,s,t)$$

with a bounded function K'.

Since our functions are positive, in order to prove (3.14) it is enough to estimate (3.15) for, say, $t > 1$. In that case $(\sinh t)^{-1} \leq \frac{1}{4} e^{-t}$; using also that $\sinh r < \cosh r, \frac{1}{2} e^r < \cosh r < e^r$, we see that (3.15) is majorized by

$$C e^{-\rho t} \int \int_{|r-s| \leq t \leq r+s} e^{-(\rho+2)r - (\rho+2+2k)s} dr ds.$$

The integral is easy to compute by the variable change $u = r + s, v = r - s$, and gives a majorization by $C' e^{-2(\rho+1)t}$, which implies (3.14).

The next important fact is that the T_k's all commute and that the identity

$$\Delta T_k = 4(\rho + k + 1)(k + 1)(T_k - T_{k+1}) \tag{3.16}$$

holds. Since the radial functions on a symmetric space form a commutative convolution algebra and Δ commutes with radial convolutions, for this it suffices to prove that

$$\Delta h_k = 4(\rho + k + 1)(k + 1)(h_k - h_{k+1}).$$

This is an easy computation using (3.8).

We only sketch the rest of the proof which contains nothing new. Repeated application of (3.16) gives

$$T_k f = G_{k,\rho}(\Delta) T_0 f \tag{3.17}$$

with the polynomial

$$G_{k,\rho}(\mu) = \prod_{j=1}^{k} (1 - \frac{\mu}{4j(\rho + j)}). \tag{3.18}$$

The subspace $M = \{f \in L^1(h_0 dv) | T_0 f = f\}$ is closed in $L^1(h_0 dv)$; this can be seen by observing that by (3.14) T_0 is a bounded operator from $L^1(h_0 dv)$ to $L^1(h_1 dv)$. The operator Δ preserves M, and its restriction, to be denoted by Δ_M, is bounded on M by (3.16). Writing $G_\rho(\mu)$ for the infinite product obtained from (3.18) by letting $k \to \infty$, it follows that

$$G_\rho(\Delta_M) = I_M, \tag{3.19}$$

the identity map.

By the product expansion of the gamma function we have $G_\rho(-(\lambda^2 + \rho^2)) = \hat{h}_0(\lambda)^{-1}$. We write Ω_ρ for the image of the strip $|Im\, \lambda| < \rho + 2$ under the map $\lambda \mapsto -(\lambda^2 + \rho^2)$ and E_ρ for the subset of Ω_ρ where $G_\rho(\mu) = 1$. By the remark after the proof of Proposition 3.1, E_ρ is exactly the point spectrum of Δ_M.

Writing Q for the monic polynomial whose zeroes are the points of E_ρ, there exists an entire function H such that $HQ = G_\rho - 1$, and we have $H(\mu) \neq 0$ at every $\mu \in E_\rho$. Since H has no zero on the point spectrum of Δ_M, by the spectral mapping theorem 0 is not in the point spectrum of $H(\Delta_M)$, which is therefore a one-to-one operator. (3.19) shows that $H(\Delta_M)Q(\Delta_M) = 0$, and it follows that $Q(\Delta_M) = 0$. [1, Lemma 4.1] can now be applied and shows that M is the (finite) direct sum of the eigenspaces of Δ_M. By our Proposition 3.1, this finishes the proof.

REMARK. The same methods work if one wishes to study the solutions of $T_k f = f$ for some $k \neq 0$, or if one looks for solutions in some $L^p (1 < p < \infty)$ instead of L^1. In particular, all the statements of the main theorem in [1] remain true after replacing n by ρ.

4. Further Generalizations

In [3] E. Damek and F. Ricci construct a large family of examples of harmonic spaces which all have natural realizations as unit balls in some $\mathbb{R}^m$. They contain the hyperbolic spaces as special cases. As Z. Szabó [12] proved, on any harmonic space the integral kernels $K(p,q)$ depending only on the distance of p and q form a commutative algebra under composition; also, the Laplacian Δ commutes with the natural radialization operator. On the Damek-Ricci examples there acts, in addition, a simply transitive group of isometries, and the commutative algebra of kernels appears as the convolution algebra of radial functions on this group.

These examples are still characterized by two non-negative integers p and q as in the symmetric case; the dimension is still $m = p + q + 1$ and the invariant $\rho = \frac{p}{2} + q$ (denoted in [3] by Q, and known to be an integer when $q \neq 0$) still plays a crucial role. The geodesics through 0 are the diameters parametrized by $\tanh t$, the Riemannian volume element and the spherical functions have the same expressions in terms of p and q as in the symmetric case.

As long as ρ is an integer, our results apply unchanged to the Damek-Ricci spaces. The proofs are the same, with averaging over K replaced by the radialization operator (denoted by π in [3]), and with somewhat more attention paid to writing convolutions since the underlying group is not unimodular.

The number ρ is only a half-integer when $q = 0$ and m is even. This is the case of the Cayley-Klein model of m-dimensional real hyperbolic space which we excluded from consideration in section 3. Actually, our arguments still work, but in order to find for exactly what values of ρ the equation $T_0 f = f$ will have non-harmonic solution, it is necessary to make a study of the function G_ρ analogous to Propositions 3.7 and 3.8 in [1] for non-integer ρ.

Another generalization of our results is possible in the context of the "Jacobi hypergroups" introduced by M. Flensted-Jensen and T. Koornwinder [5],[6],[10]. Here there is still a (generalized) convolution algebra, characterized by two arbitrary positive numbers p and q, and a differential operator generalizing Δ. Our general considerations remain true, and the question about the existence of non-trivial solutions of $T_0 f = f$ reduces to the study of the function G_ρ as before.

References

1. P. Ahern, M. Flores and W. Rudin, *An invariant volume-mean-value property*, J. Funct. Anal. **111** (1993), 380-397.
2. M. Cowling, A.H. Dooley, A. Korányi and F. Ricci, *H-type groups and Iwasawa decompositions*, Advances in Math. **87** (1991), 1-41.
3. E. Damek and F. Ricci, *Harmonic analysis on solvable extensions of H-type groups*, J. of Geom. Anal. **2** (1992), 213-248.
4. A. Erdélyi et.al., *Higher Transcendental Functions*, McGraw-Hill, New York, 1953.
5. M. Flensted-Jensen, *Paley-Wiener type theorems for a differential operator connected with symmetric spaces*, Ark. Mat. **10** (1972), 143-162.
6. M. Flensted-Jensen and T. H. Koornwinder, *The convolution structure for Jacobi function expansions*, Ark. Mat. **10** (1973), 245-262.

7. H. Furstenberg, *A Poisson formula for semi-simple Lie groups*, Ann. Math. **77** (1963), 335-386.
8. Harish-Chandra, *Spherical functions on a semisimple Lie group, I*, Amer. J. Math. **80** (1958), 241-310.
9. S. Helgason, *Groups and Geometric Analysis*, Academic Press, New York, 1984.
10. T.H. Koornwinder, *Jacobi functions and analysis on noncompact semisimple Lie groups*, In: *Special Functions: Group Theoretical Aspects and Applications*, (R.A. Askey, et.al., eds.), 1-85, D. Reidel, Amsterdam, 1984..
11. G.D. Mostow, *Strong Rigidity of Locally Symmetric Spaces*, Ann. of Math. Studies, No. 78, Princeton Univ. Press, Princeton, NJ, 1973.
12. Z. Szabó, *The Lichnerowicz conjecture on harmonic manifolds*, J. Diff. Geom. **31** (1990), 1-28.
13. M. Takeuchi, *Cell decompositions and Morse inequalities on certain symmetric spaces*, J. Faculty of Sci. Univ. Tokyo, Ser. I. **12** (1965), 81-192.

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Inversion of an Integral Transform and Ladder Representations of $U(1,q)$

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ABSTRACT. An integral transform for $G = U(1,q)$ is studied. The transform maps the positive spin ladder representations of G on a Bargmann-Segal-Fock space $\mathcal{F}_n^{1,q}$ into a space of polynomial-valued functions on the bounded realization $\mathbf{B}^q$ of G/K. An inversion is given for the transform and unitary structures are given for the geometric realization of the positive spin ladder representations over G/K.

0. Introduction

The study of representation theory of linear reductive Lie groups is intimately related to the study of differential equations and the physics that these equations model. Such are the beginnings of the theory concerning the ladder representations of $U(p,q)$ given in this paper.

In [7], Jakobsen and Vergne show the unitarity of a cleverly constructed multiplier representation π_s of $U(2,2)$, the 4-1 cover of the conformal group, on a Hilbert space $\mathcal{H}_s$ of holomorphic functions on $U(2,2)/K$ realized as the generalized upper half plane. The elements of $\mathcal{H}_s$ satisfy the mass 0, spin s equations, and the restriction of π_s to the Poincaré group is the representation of mass 0, spin s.

A previous result that is recurrent throughout [7] is a theorem of Kunze (see [9]) that guarantees the unitarity of certain multiplier representations. The

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key is that these representation spaces of functions possess a positive definite reproducing kernel that behaves well with respect to the action of the maximal compact subgroup on the image space of these functions.

Meanwhile, the oscillator representation of $U(p,q)$ has been studied and decomposed into irreducible ladder representations in [**2**], [**3**], [**8**], and [**16**] to name a few. Consider the realization in [**3**]. Here the oscillator representation σ is constructed via the Stone-von Neumann theorem on a Fock space $\mathcal{F}$ of (p,q)-holomorphic functions. We use this realization of σ in this paper, despite the geometric advantages of the realization given in [**2**].

The relationship between the representations π_s and the oscillator representation of $U(2,2)$ noted in [**7**] is generalized by Mantini in [**10**]. It is in [**10**] that we see the positive spin ladder representations $\mathcal{F}_n$ of $U(p,q)$ realized via an integral transform Φ_n as a subspace $\mathcal{S}_n$ of sections of a homogeneous holomorphic vector bundle $\mathbf{E}_n$ over $U(p,q)/K$ in its model as a generalized unit disk $\mathbf{D}$. The fiber of $\mathbf{E}_n$ over any point $\zeta \in \mathbf{D}$ is essentially the space $\bar{\mathcal{P}}(n, \mathbf{C}^q)$ of antiholomorphic polynomials homogeneous of degree n in q complex variables. If the real rank of $U(p,q)$ is larger than one, or if the rank of $\mathbf{E}_n$ is larger than one, these sections satisfy certain linear partial differential equations. In the $U(2,2)$ case, these equations are the massless field equations and the natural action on $\mathcal{S}_n$ is equivalent to the multiplier representation of [**7**]. The transform Φ_n bears many similarities to the Penrose transform (see [**12**] and [**17**]). We will discuss a version of Φ_n more carefully in section 2 below. There also exists an analogous construction of the negative spin ladder representations (see [**11**]), but here we restrict to the positive spin case.

Since Φ_n is one-to-one on $\mathcal{F}_n$ for $n \geq 0$, Φ_n is invertible. In section 3, an inverse for Φ_n in the case when $G = U(1,q)$ is presented. Trailing along behind the inversion of Φ_n is the machinery that allows the production of a Hilbert space structure on $\mathcal{S}_n$ so that $\mathcal{S}_n$ is unitarily equivalent to $\mathcal{F}_n$ via Φ_n. Such structures are exhibited in section 4 for this case.

Motivation for our results is essentially two-fold. On one hand, we consider the results of Davidson [**3**] (which again depend critically upon results of Kunze [**9**]). He gives natural unitary norms on $U(p,q)/K$ for discrete series representations of $U(p,q)$. These structures take on the form

$$(f_1, f_2) = \int_{\mathbf{D}} \langle Q_\lambda(\zeta,\zeta)^{-1} f_1(\zeta), f_2(\zeta) \rangle \, d\mu(\zeta),$$

where $f_i \in \mathcal{H}(\mathbf{D}, \lambda)$ are polynomial-valued functions on the disk, $d\mu$ is the $U(p,q)$-invariant measure on $\mathbf{D}$, and Q_λ is the reproducing kernel for $\mathcal{H}(\mathbf{D}, \lambda)$. The inner product $\langle \cdot, \cdot \rangle$ consists of an integral with respect to the appropriate Gaussian measure. For the ladder representations of $U(p,q)$ which are not square-integrable, an analogous norm does not converge (see Lemma 4.2 of [**3**]).

On the other hand, we are also inspired by inversion formulas for the Radon transform, as given in [**5**]. Let $\mathbf{P}^n$ denote the space of all hyperplanes in $\mathbf{R}^n$, and let f be a function on $\mathbf{R}^n$ that is integrable on each element ξ of $\mathbf{P}^n$ with

respect to Euclidean measure dm_ξ on ξ. The Radon transform $\hat{f}$ of f is then given by

$$\hat{f}(\xi) = \int_\xi f(x)\, dm_\xi(x),$$

while the dual transform for continuous functions φ on $\mathbf{P}^n$ is given by

$$\check{\varphi}(x) = \int_{\xi_x} \varphi(\xi)\, d\mu_x(\xi),$$

where $\xi_x = \{\xi \in \mathbf{P}^n \mid x \in \xi\}$ and $d\mu_x$ is the measure of total mass one on ξ_x that is invariant under rotation about x. A Schwartz function f can be recovered from its Radon transform via the formula

$$cf = \mathcal{L}^{\frac{n-1}{2}}((\hat{f})\check{}),$$

where c is a constant and $\mathcal{L}$ is the Laplacian on $\mathbf{R}^n$. If n is even, $\mathcal{L}^{\frac{n-1}{2}}$ is defined in terms of Riesz potentials (see Lemma 2.35 of [5]). Operators playing a role similar to these fractional powers of the Laplacian appear in our results.

In large part, we achieve our results by constructing two special operators on $\mathcal{O}(\mathbf{D}, \bar{\mathcal{P}}(n, \mathbf{C}^q))$, P and L. Let ϕ be the image of a monomial in $\mathcal{F}_n$ under Φ_n. Fixing $\zeta \in \mathbf{D}$ and considering $\phi(\zeta)$ as an element of $\bar{\mathcal{P}}(n, \mathbf{C}^q)$, the mapping P is a projection operator such that $P\phi(\zeta)$ is the nonzero summand of $\phi(\zeta)$ of highest possible weight with respect to the K action on $\bar{\mathcal{P}}(n, \mathbf{C}^q)$. The mapping L is simply a weighted version of the mapping P which enables the overall inversion and inner products to become unitary. Using this mapping L, we produce an inversion formula for Φ_n and the desired unitary structures. These have the form

$$(\Phi_n^{-1}\phi)(z) = \lim_{t \to 1^-} \frac{d^q}{dt^q}\left[t^q \int_{\mathbf{D}} \int_{\mathbf{C}^q} (L\phi)(t\zeta, v) e^{(z_1\bar{\zeta} + v^T)\bar{z}_S}\, e^{-|v|^2}\, dm(v) dm(\zeta)\right],$$

and

$$(\!(\phi_1, \phi_2)\!) = \lim_{t \to 1^-} \frac{d^q}{dt^q}\left[t^q \int_{\mathbf{D}} \langle (L\phi_1)(t\zeta, \cdot), \phi_2(\zeta, \cdot)\rangle\, dm(\zeta)\right],$$

respectively.

Alternative unitary structures for most of the ladder representations of $U(p, q)$ are given by Rawnsley, Schmid, and Wolf (see section 13 of [14]). These representations occur in L^2-cohomology of certain homogeneous, holomorphic vector bundles over $U(p, q)/(U(1) \times U(p-1, q))$.

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1. Preliminaries

In this section we wish to gain familiarity with the oscillator representation of $U(p, q)$ realized on a certain Fock space. The generalized unit disk and an

associated representation will be defined. We first introduce the multi-index notation that will be prevalent throughout the paper.

1.1. NOTATION.

(a) For $r \in \mathbf{N}$, let $\mathbf{N}_0$ denote the set of nonnegative integers and let $\mathbf{N}_0^r$ denote the set of r-tuples of nonnegative integers.

(b) If $m = (m_1, m_2, \ldots, m_r) \in \mathbf{N}_0^r$, then $m! := m_1! m_2! \cdots m_r!$ and $|m| := m_1 + m_2 + \cdots + m_r$. If $z \in \mathbf{C}^r$, then $z^m := z_1^{m_1} z_2^{m_2} \cdots z_r^{m_r}$.

(c) Let $\alpha \in \mathbf{N}_0^s$, $1 \leq s \leq r$, and let $J = (j_1, j_2, \ldots, j_s) \in \mathbf{N}_0^s$ be such that $1 \leq j_1 < j_2 < \cdots < j_s \leq r$. Then, if $z \in \mathbf{C}^r$, $z_J := (z_{j_1}, z_{j_2}, \ldots, z_{j_s})$, $|z_J|^2 := |z_{j_1}|^2 + |z_{j_2}|^2 + \cdots + |z_{j_s}|^2$, and $z_J^\alpha := z_{j_1}^{\alpha_1} z_{j_2}^{\alpha_2} \cdots z_{j_s}^{\alpha_s}$.

(d) Let $R_p := (1, 2, \ldots, p)$ and $S_{p,q} := (p+1, p+2, \ldots, p+q)$. When there is no danger of confusion, we will let R and S represent R_p and $S_{p,q}$, respectively.

(e) For $n \in \mathbf{N}_0$ and $q \in \mathbf{N}$, let $\mathbf{N}_0^q(n) := \{\eta \in \mathbf{N}_0^q \mid |\eta| = n\}$, and let $d(n, q)$ denote the cardinality of $\mathbf{N}_0^q(n)$. Note that $d(n, q) = \dim \bar{\mathcal{P}}(n, \mathbf{C}^q)$.

(f) The symbol $\prec$ will denote lexicographic, or dictionary, order on $\mathbf{N}_0^q(n)$. Here $\alpha \prec \beta$ if there exists $j \in \{1, \ldots, q\}$ such that $\alpha_i = \beta_i$ for $i < j$ and $\alpha_j < \beta_j$. In this case we say that β succeeds α.

(g) We have a partial order $\leq$ on $\mathbf{N}_0^q$ given by $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for $i \in \{1, \ldots, q\}$.

(h) Let $\epsilon_j^q := (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbf{N}_0^q$, where the support lies in the j-th component of ϵ_j^q. We will write ϵ_j for ϵ_j^q when there is no danger of confusion.

(i) For $p, q \in \mathbf{N}$, we let $M^{p \times q}(\mathbf{C})$ denote the set of $p \times q$ matrices with complex entries. If $X \in M^{p \times q}(\mathbf{C})$, the conjugate transpose of X is written as X^*.

Recall that for $p, q \in \mathbf{N}$, $U(p, q) = \{g \in GL(p+q, \mathbf{C}) \mid g^* I_{p,q} g = I_{p,q}\}$, where $I_{p,q}$ is the element of $M^{(p+q)\times(p+q)}(\mathbf{C})$ given by

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

If $g \in G = U(p, q)$, the matrix g is occasionally written in block form as

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A and D are $p \times p$ and $q \times q$ matrices, respectively. In addition, the subgroup $K \simeq U(p) \times U(q)$ of $U(p, q)$ given by

$$K := \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \mid A \in U(p),\ D \in U(q) \right\}$$

is a maximal compact subgroup.

Let $dm(z)$ denote the normalization of Lebesgue measure on $\mathbf{C}^r$ given by

$$dm(z) := \frac{1}{\pi^r} dx_1 dy_1 \dots dx_r dy_r.$$

The following lemmas will be useful.

1.2. LEMMA. *Let $k, l \in \mathbf{N}_0$, and let $\delta_{kl} = 1$ if $k = l$ and $\delta_{kl} = 0$ otherwise. Then*

$$\int_{\mathbf{C}} z^k \bar{z}^l e^{-|z|^2}\, dm(z) = \delta_{kl}\, k!.$$

1.3. LEMMA. *Suppose that $U \in \mathbf{C}^{q\times q}$ is such that $U + U^* \gg 0$, and ϕ is a holomorphic function of $z \in \mathbf{C}^q$ for which $\int \phi(z) \exp(-z^* U z)\, dm(z)$ converges absolutely. Then*

$$\int_{\mathbf{C}^q} \phi(z) e^{-z^* U z}\, dm(z) = \frac{\phi(0)}{\det U}.$$

1.4. LEMMA. *Suppose that $q \in \mathbf{N}$ and $w \in \mathbf{C}$. The series*

$$\sum_{m \in \mathbf{N}_0^q} \frac{w^{|m|}}{|m|^q}$$

converges if $|w| < 1$ and diverges if $|w| > 1$.

Finally, we say that a complex-valued function f on $\mathbf{C}^{p+q}$ is *(p,q)-holomorphic* if f is holomorphic in z_R and antiholomorphic in z_S.

1.1. The Oscillator Representation of $U(p, q)$. Consider the Bargmann-Segal-Fock space

$$\mathcal{F}^{p,q} = \{f : \mathbf{C}^{p+q} \to \mathbf{C} \mid f \text{ is } (p,q)\text{-holomorphic and } \int_{\mathbf{C}^{p+q}} |f(z)|^2 e^{-|z|^2} dm(z) < \infty\}. \tag{1.1}$$

By polarizing the integral in (1.1), one obtains the inner product for the Hilbert space $\mathcal{F}^{p,q}$, which will be denoted by $\langle \cdot, \cdot \rangle$. Let $f \in \mathcal{F}^{p,q}$ and suppose

$$f(z) = \sum_{\substack{l \in \mathbf{N}_0^p \\ m \in \mathbf{N}_0^q}} a_{l,m}\, z_R^l\, \bar{z}_S^m.$$

It follows from Lemma 1.2 that

$$\langle f, f \rangle = \sum_{\substack{l \in \mathbf{N}_0^p \\ m \in \mathbf{N}_0^q}} |a_{l,m}|^2\, l!\, m!. \tag{1.2}$$

As an application of Lemma 1.3, we obtain the reproducing formula for $f \in \mathcal{F}^{p,q}$ given by

$$f(z) = \int_{\mathbf{C}^{p+q}} f(w) e^{w_R^* z_R + z_S^* w_S} e^{-|w|^2}\, dm(w). \tag{1.3}$$

We now define a version of the oscillator representation σ of G.

1.5. DEFINITION. For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$ and $f \in \mathcal{F}^{p,q}$, define $\sigma(g)f$ by

$$[\sigma(g)f](z) = \det(D^*) \int_{\mathbf{C}^{p+q}} f(g^{-1}w) e^{-\frac{1}{2}|g^{-1}w|^2} e^{w_R^* z_R + z_S^* w_S} e^{-\frac{1}{2}|w|^2} dm(w).$$

It is not immediate that σ defines a unitary representation of G on $\mathcal{F}^{p,q}$. Nonetheless we have the following theorem, due to Blattner and Rawnsley.

1.6. THEOREM [**2**]. *The map σ of Definition 1.5 is a unitary representation of G on $\mathcal{F}^{p,q}$.*

Now, for $n \in \mathbf{Z}$, let

$$\mathcal{F}_n^{p,q} := \{f \in \mathcal{F}^{p,q} \mid f(e^{-i\theta}z) = e^{in\theta} f(z), \theta \in \mathbf{R}\}.$$

A short computation shows that $\mathcal{F}_n^{p,q}$ is a G-invariant subspace of $\mathcal{F}^{p,q}$. For $g \in G$, let $\sigma_n(g)$ denote the restriction of $\sigma(g)$ to $\mathcal{F}_n^{p,q}$.

1.7. THEOREM [**16**]. *Let $n \in \mathbf{Z}$. The representation σ_n of G on $\mathcal{F}_n^{p,q}$ is irreducible, and there is an orthogonal direct sum decomposition*

$$\mathcal{F}^{p,q} = \bigoplus_{n \in \mathbf{Z}} \mathcal{F}_n^{p,q}.$$

REMARK. The representations σ_n are often referred to as *ladder* representations, since the highest weights of their K-types lie equally spaced along a ray in the weight lattice.

1.2. The Generalized Unit Disk. For $p, q \in \mathbf{N}$, let $\mathbf{C}^{p,q}$ denote $\mathbf{C}^{p+q}$ endowed with the hermitian form h with matrix $I_{p,q}$. A subspace V of $\mathbf{C}^{p,q}$ is *negative* if the restriction of h to V is negative definite. Let $\mathbf{M}_{p,q}^-$ denote the set of all negative q-planes in $\mathbf{C}^{p,q}$. The *generalized unit disk* $\mathbf{D}_{p,q}$ is given by

$$\mathbf{D}_{p,q} = \{\zeta \in M^{p\times q}(\mathbf{C}) \mid I_q - \zeta^*\zeta \gg 0\}. \tag{1.4}$$

1.8. PROPOSITION. *The mapping $\zeta \mapsto V(\zeta)$ of $\mathbf{D}_{p,q}$ into $\mathbf{M}_{p,q}^-$ where*

$$V(\zeta) = \text{col span} \begin{bmatrix} \zeta \\ I_q \end{bmatrix}$$

gives a parameterization of $\mathbf{M}_{p,q}^-$ by $\mathbf{D}_{p,q}$.

PROOF. The mapping is clearly well-defined and injective into the space of q-planes. The q-plane $V(\zeta)$ associated to $\zeta \in \mathbf{D}_{p,q}$ is negative due to the positivity condition in (1.4). It remains to show that the mapping is surjective. Let

$$V = \text{col span} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \in \mathbf{M}_{p,q}^-,$$

where U_1 and U_2 are $p \times q$ and $q \times q$ matrices, respectively. Now U_2 must be nonsingular, as otherwise V will admit positive vectors. Hence we can change

basis to see that $V = V(U_1U_2^{-1})$. Since V is negative, $U_1U_2^{-1}$ satisfies the conditions of (1.4), therefore $U_1U_2^{-1} \in \mathbf{D}_{p,q}$. ■

The natural left action of G on $\mathbf{M}_{p,q}^-$ given by

$$g.V(\zeta) = \text{col span}\left(g\begin{bmatrix}\zeta\\ I_q\end{bmatrix}\right)$$

is transitive, and the stabilizer of $V(0)$ is K. By Proposition 1.8, $\mathbf{D}_{p,q}$ is diffeomorphic to G/K and there is a natural action of G on $\mathbf{D}_{p,q}$ that is stabilized at $0 \in \mathbf{D}_{p,q}$ by K. This action is given by

$$g.\zeta = (A\zeta + B)(C\zeta + D)^{-1},$$

where $g = \begin{pmatrix}A & B\\ C & D\end{pmatrix} \in G$ and $\zeta \in \mathbf{D}_{p,q}$.

Note that $\mathbf{D}_{p,q}$ is a bounded domain in $\mathbf{C}^{pq}$. In fact, it is one of the four *classical domains* (cf. [**6**] or [**13**]). In the case we are concerned with, that is $p = 1$, $\mathbf{D}_{1,q}$ is simply the unit ball $\mathbf{B}^q$ in $\mathbf{C}^q$.

We will rely on our ability to integrate over $\mathbf{B}^q$, and toward that end we use the following result. For details, see [**15**].

1.9. LEMMA. *Let $\eta, \gamma \in \mathbf{N}_0^q$ and suppose that $|\eta| = n$ and $|\gamma| = m$. Then*

$$\int_{\mathbf{B}^q} \zeta^\eta \bar{\zeta}^\gamma dm(\zeta) = \delta_{\eta,\gamma}\left(\frac{\eta!}{(n+q)!}\right),$$

where $\delta_{\eta,\gamma} = 1$ if $\eta = \gamma$ and $\delta_{\eta,\gamma} = 0$ otherwise.

Let $\bar{\mathcal{P}}(n, \mathbf{C}^q)$ denote the set of antiholomorphic polynomials on $\mathbf{C}^q$ that are homogeneous of degree n and let $\mathcal{O}(\mathbf{D}_{p,q}, \bar{\mathcal{P}}(n, \mathbf{C}^q))$ denote the set of functions holomorphic on $\mathbf{D}_{p,q}$ taking values in $\bar{\mathcal{P}}(n, \mathbf{C}^q)$. We conclude this section with the construction of a representation ω_n of G on $\mathcal{O}(\mathbf{D}_{p,q}, \bar{\mathcal{P}}(n, \mathbf{C}^q))$. Define, for $n \geq 0$, a mapping $J_n : G \times \mathbf{D}_{p,q} \to GL(\bar{\mathcal{P}}(n, \mathbf{C}^q))$ by

$$J_n(g, \zeta)f(v) = \det[C\zeta + D]f([C\zeta + D]^* v) \tag{1.5}$$

for $g = \begin{pmatrix}A & B\\ C & D\end{pmatrix}$, $\zeta \in \mathbf{D}_{p,q}$ and $v \in \mathbf{C}^q$. It is easily shown that J_n is a *factor of automorphy*. That is, if $g_1, g_2 \in G$ and $\zeta \in \mathbf{D}_{p,q}$ then $J_n(I, \zeta) = \text{Id}$ and $J_n(g_1g_2, \zeta) = J_n(g_1, g_2.\zeta) \circ J_n(g_2, \zeta)$.

1.10. DEFINITION. Suppose $g \in G$ and $\phi \in \mathcal{O}(\mathbf{D}_{p,q}, \bar{\mathcal{P}}(n, \mathbf{C}^q))$. Define $\omega_n(g)\phi$ by

$$(\omega_n(g)\phi)(\zeta, v) = J_n(g^{-1}, \zeta)^{-1}\phi(g^{-1}.\zeta, v)$$

for $\zeta \in \mathbf{D}_{p,q}$ and $v \in \mathbf{C}^q$.

Note that ω_n is a multiplier representation of the type discussed in [**9**]. This representation corresponds to the natural geometric action of G on the vector bundle $\mathbf{E}_n$. In the next section we will intertwine σ_n and ω_n via an integral transform.

2. The Integral Transform

In [**10**], Mantini constructs an integral transform Φ_n for each $n \in \mathbf{Z}$, $n \geq 0$. We may think of Φ_n as a mapping of $\mathcal{F}_n^{p,q}$ into $\mathcal{O}(\mathbf{D}_{p,q}, \bar{\mathcal{P}}(n, \mathbf{C}^q))$. This transform is analogous to the Penrose transform (see [**12**]). In case either the real rank of G is larger than one or the dimension of $\bar{\mathcal{P}}(n, \mathbf{C}^q)$ is larger than one (i.e., when $n > 0$ and $q > 1$), the elements in the image of Φ_n satisfy certain linear partial differential equations. These equations generalize the massless field equations of mathematical physics which hold when $G = U(2,2)$ (see [**10**] or [**7**]). The image of the map Φ_n in [**10**] is in a space of sections of a holomorphic, homogeneous vector bundle on which the group acts by the natural, geometric action. Thus Φ_n implicitly gives a geometric construction of the ladder representations σ_n, although the inner product structure on this realization has previously been unknown.

At this point let us recall that our primary goal is to create an inversion for Φ_n and then use this inversion to develop unitary structures for the geometric realization of σ_n obtained via Φ_n. In this section we familiarize ourselves with Φ_n and some of its important properties. Further details may be found in [**10**] and [**11**].

2.1. PROPOSITION [**10**]. *Suppose $n \geq 0$ and $f \in \mathcal{F}_n^{p,q}$. Then we have an integral transform $\Phi_n : \mathcal{F}_n^{p,q} \to \mathcal{O}(\mathbf{D}_{p,q}, \bar{\mathcal{P}}(n, \mathbf{C}^q))$ given by*

$$(\Phi_n f)(\zeta, v) = \int_{\mathbf{C}^q} f(\zeta w, w) e^{v^* w} e^{-|w|^2}\, dm(w),$$

where $v \in \mathbf{C}^q$ and $(\Phi_n f)(\zeta, v)$ depends holomorphically on $\zeta \in \mathbf{D}_{p,q}$.

The transform of Proposition 2.1 is simplified from the transform constructed by Mantini in [**10**]. Here we have used the trivialization of the vector bundle in [**10**] with the multiplier action ω_n. We have also used Davidson's model of the Fock space representation σ_n instead of the L^2-cohomology realization of Blattner and Rawnsley (see [**2**]). The transform is essentially given by restriction to a negative q-plane determined by $\zeta \in \mathbf{D}_{p,q}$ followed by a projection operator, giving rise to an L^2 version of the Penrose transform. Notice that the projection operator is the specialization to signature $(0, q)$ of the reproducing formula given in (1.3).

The first important feature of Φ_n is that it serves as an intertwining map between the representations σ_n and ω_n.

2.2. THEOREM [**10**]. *The mapping Φ_n is one-to-one for $n \geq 0$ and identically zero for $n < 0$. Furthermore, for all $f \in \mathcal{F}_n^{p,q}$, $g \in G$, and $n \geq 0$, the mapping Φ_n satisfies*

$$\omega_n(g)(\Phi_n f) = \Phi_n(\sigma_n(g) f).$$

Theorem 2.2 implies that for $n \geq 0$, Φ_n has an inverse defined on $\Phi_n(\mathcal{F}_n^{p,q})$. Also, it says that σ_n is realized on $\Phi_n(\mathcal{F}_n^{p,q})$ by a subrepresentation of ω_n.

2.3. LEMMA [10]. *Fix $n \in \mathbf{Z}$ and $\zeta \in \mathbf{D}_{p,q}$. The mapping of $\mathcal{F}_n^{p,q}$ into $\mathcal{F}_n^{0,q}$ given by*

$$f \mapsto \Phi_n f(\zeta, \cdot)$$

is continuous.

The next thing to note is that for many values of p and q, the elements of $\Phi_n(\mathcal{F}_n^{p,q})$ satisfy certain interesting differential equations. Using the notation of [4], let $\square_{p,q}$ be the $(p+1) \times q$ matrix of differential operators where

$$(\square_{p,q})_{ij} = \begin{cases} \frac{\partial}{\partial \zeta_{ij}}, & \text{for } 1 \le i \le p \text{ and } 1 \le j \le q, \\ \frac{\partial}{\partial \bar{v}_j}, & \text{for } i = p+1 \text{ and } 1 \le j \le q. \end{cases}$$

2.4. THEOREM [10].

(a) *Let $p, q \in \mathbf{N}$ be such that both p and q are greater than one. Then $\Phi_n(\mathcal{F}_n^{p,q})$ is a subset of $\mathcal{O}(\mathbf{D}_{p,q}, \bar{\mathcal{P}}(n, \mathbf{C}^q))$ annihilated by the differential operators formed by taking 2×2-minors of $\square_{p,q}$.*

(b) *If $p = 1$, $n \ge 1$ and $q > 1$, then $\Phi_n(\mathcal{F}_n^{1,q})$ is a subset of $\mathcal{O}(\mathbf{B}^q, \bar{\mathcal{P}}(n, \mathbf{C}^q))$ that is annihilated by the differential operators formed by taking 2×2-minors of $\square_{1,q}$.*

REMARK. In case $p = q = 2$, the differential operators formed by taking 2×2-minors of $\square_{2,2}$ are the complexifications of the massless field equations. Also, it has been shown in [4] that the K-finite vectors in $\mathcal{O}(\mathbf{D}_{p,q}, \bar{\mathcal{P}}(n, \mathbf{C}^q))$ with respect to ω_n that satisfy the differential equations mentioned in the theorem are exactly the K-finite vectors in $\Phi_n(\mathcal{F}_n^{p,q})$.

Suppose that $n \in \mathbf{N}_0$ and $\phi \in \mathcal{O}(\mathbf{D}_{p,q}, \bar{\mathcal{P}}(n, \mathbf{C}^q))$. Then there exists, for each $\eta \in \mathbf{N}_0^q(n)$, a holomorphic complex-valued function $\psi^{(\eta)}$ on $\mathbf{D}_{p,q}$ such that

$$\phi(\zeta, v) = \sum_{\eta \in \mathbf{N}_0^q(n)} \psi^{(\eta)}(\zeta) \frac{\bar{v}^\eta}{\eta!}. \tag{2.1}$$

The component functions $\psi^{(\eta)}$ in (2.1) are indexed by $\eta \in \mathbf{N}_0^q(n)$ under the lexicographic order $\prec$ (see part (f) of Notation 1.1), with indices ranging in order from $(0, \ldots, 0, n)$ through $(n, 0, \ldots, 0)$. In what follows, this description of $\phi(\zeta, v)$ will sometimes be convenient.

We finish this section with a lemma that will be of importance later on. Define, for $\alpha \in \mathbf{N}_0^p$, $\beta \in \mathbf{N}_0^q$ and $z \in \mathbf{C}^{p+q}$,

$$f_{\alpha\beta}(z) := z_R^\alpha \bar{z}_S^\beta. \tag{2.2}$$

We wish to compute the image of $f_{\alpha\beta}$ under the transform $\Phi_{(|\beta|-|\alpha|)}$. In order to simplify our expressions, we first introduce some notation.

Given $\gamma \in M^{p \times q}(\mathbf{C})$ and $i \in \{1, 2, \ldots, p\}$, let $\gamma_{(i)}$ denote the i-th row of γ. Let $r(\gamma)$ denote the element of $\mathbf{C}^q$ obtained by taking the sum of the rows of

γ, and let $c(\gamma) \in \mathbf{C}^p$ denote the sum of the columns of γ. Given $\alpha \in \mathbf{N}_0^p$ and $\tilde{\alpha} \in \mathbf{N}_0^q$, let $M^{p\times q}(\alpha, \tilde{\alpha})$ denote the subset of $M^{p\times q}(\mathbf{N}_0)$ whose elements γ satisfy

$$c(\gamma) = \alpha \qquad \text{and} \qquad r(\gamma) = \tilde{\alpha}.$$

Notice that $M^{p\times q}(\alpha, \tilde{\alpha})$ is nonempty only when $|\alpha| = |\tilde{\alpha}|$. Now, if $\gamma \in M^{p\times q}(\mathbf{N}_0)$ and $\zeta \in \mathbf{D}_{p,q}$, we may write ζ^γ and $\gamma!$ for

$$\prod_{i=1}^{p} \zeta_{(i)}^{\gamma_{(i)}} \qquad \text{and} \qquad \prod_{i=1}^{p} \gamma_{(i)}!,$$

respectively. Finally, when p and q are understood, we let $M(\alpha, \tilde{\alpha})$ stand for $M^{p\times q}(\alpha, \tilde{\alpha})$.

2.5. LEMMA. *Suppose that $n \geq 0$, $\alpha \in \mathbf{N}_0^p$ and $\beta \in \mathbf{N}_0^q$ with $|\beta| - |\alpha| = n$. Then $f_{\alpha\beta} \in \mathcal{F}_n^{p,q}$ and*

$$\big(\Phi_n f_{\alpha\beta}\big)(\zeta, v) = \sum_{\substack{\eta \in \mathbf{N}_0^q(n) \\ \eta \leq \beta}} \sum_{\gamma \in M(\alpha, \beta-\eta)} \Big(\frac{\alpha!\,\beta!}{\gamma!}\zeta^\gamma\Big)\frac{\bar{v}^\eta}{\eta!},$$

where $\zeta \in \mathbf{D}_{p,q}$ and $v \in \mathbf{C}^q$. The order $\leq$ is as in part (g) of Notation 1.1.

PROOF. Let ϕ denote $\Phi_n f_{\alpha\beta}$. According to Proposition 2.1, we have

$$\begin{aligned}\phi(\zeta, v) &= \int_{\mathbf{C}^q} f_{\alpha\beta}(\zeta w, w) e^{v^* w} e^{-|w|^2}\, dm(w) \\ &= \int_{\mathbf{C}^q} \Big[\prod_{i=1}^{p}\Big(\sum_{j=1}^{q} \zeta_{ij} w_j\Big)^{\alpha_i}\Big] \bar{w}^\beta e^{v^* w} e^{-|w|^2}\, dm(w).\end{aligned}$$

At this point, we apply the generalized binomial theorem and obtain

$$\begin{aligned}\phi(\zeta, v) &= \int_{\mathbf{C}^q} \Big[\prod_{i=1}^{p}\Big(\sum_{\substack{\gamma_{(i)} \in \mathbf{N}_0^q \\ |\gamma_{(i)}| = \alpha_i}} \zeta_{(i)}^{\gamma_{(i)}} w^{\gamma_{(i)}} \frac{\alpha_i!}{\gamma_{(i)}!}\Big)\Big] \bar{w}^\beta e^{v^* w} e^{-|w|^2}\, dm(w) \\ &= \sum_{\substack{\gamma_{(1)}, \ldots, \gamma_{(p)} \in \mathbf{N}_0^q \\ |\gamma_{(i)}| = \alpha_i}} \frac{\alpha!\,\zeta^\gamma}{\gamma!} \int_{\mathbf{C}^q} w^{r(\gamma)} \bar{w}^\beta e^{v^* w} e^{-|w|^2}\, dm(w),\end{aligned}$$

where γ is the $p \times q$ matrix with rows $\gamma_{(1)}, \ldots, \gamma_{(p)}$.

Now, we are able to compute the integral in the preceding line by expanding the exponential function into its Taylor series and then applying Lemma 1.2 repeatedly. Doing so, we obtain

$$\phi(\zeta, v) = \sum_{\substack{\gamma \in M^{p\times q}(\mathbf{N}_0) \\ c(\gamma) = \alpha}} \frac{\alpha!\,\zeta^\gamma}{\gamma!}\, \epsilon(\beta, \gamma) \Big[\frac{\bar{v}^{\beta - r(\gamma)}\beta!}{(\beta - r(\gamma))!}\Big],$$

where $\epsilon(\beta, \gamma) = 1$ if $\beta \geq r(\gamma)$, and $\epsilon(\beta, \gamma) = 0$ if $\beta - r(\gamma)$ has any negative components. The result then follows immediately by gathering terms of the form $\bar{v}^\eta/\eta!$ for each $\eta \in \mathbf{N}_0^q(n)$. ∎

2.6. COROLLARY. *Fix $p = 1$, $n \geq 0$ and $\nu \in \mathbf{N}_0^q$ with $|\nu| \geq n$. Let $f_{\nu,n} \in \mathcal{F}_n^{1,q}$ be defined by*

$$f_{\nu,n}(z) := z_1^{|\nu|-n} \bar{z}_S^{\nu},$$

and let $\phi_\nu = \Phi_n(f_{\nu,n})$. Then

$$\phi_\nu(\zeta, v) = \sum_{\substack{\eta \in \mathbf{N}_0^q(n) \\ \eta \leq \nu}} \left(\frac{\nu!\,(|\nu|-n)!}{(\nu-\eta)!}\, \zeta^{\nu-\eta} \right) \frac{\bar{v}^\eta}{\eta!}.$$

3. An Inversion Formula

The goal of this section is to give an inversion formula for the integral transform Φ_n of section 2, when $n \geq 0$, $p = 1$, and $q \geq 1$. Throughout this section, p, q, and n will remain as in the previous sentence. We begin with the introduction of two mappings of $\mathcal{O}(\mathbf{B}^q, \bar{\mathcal{P}}(n, \mathbf{C}^q))$ into itself. One mapping, P, is a projection that isolates a nonzero summand of $\Phi_n f_{\nu,n}$. The other mapping, L, is simply a weighted version of P which we then use to invert the transform Φ_n. The weighting of L is chosen to ensure the G-equivariance of our inversion formula.

Let $\nu \in \mathbf{N}_0^q$ with $|\nu| \geq n$ and let $f_{\nu,n}(z) = z_1^{|\nu|-n} \bar{z}_S^{\nu}$, as in Corollary 2.6. For each $\eta \in \mathbf{N}_0^q(n)$ we define a complex-valued function $\psi_\nu^{(\eta)}$ on $\mathbf{B}^q$ so that

$$\phi_\nu(\zeta, v) = \Phi_n f_{\nu,n}(\zeta, v) = \sum_{\eta \in \mathbf{N}_0^q(n)} \psi_\nu^{(\eta)}(\zeta)\, \frac{\bar{v}^\eta}{\eta!}, \tag{3.1}$$

as in (2.1). The functions $\psi_\nu^{(\eta)}$ were calculated in Corollary 2.6.

3.1. LEMMA. *Let $\nu \in \mathbf{N}_0^q$ with $|\nu| \geq n$. Then there exists $\eta \in \mathbf{N}_0^q(n)$ such that $\psi_\nu^{(\eta)}$ is not identically zero on $\mathbf{B}^q$.*

PROOF. Since $|\nu| \geq n$, there exists $\eta \in \mathbf{N}_0^q(n)$ such that $\nu \geq \eta$. By Corollary 2.6, $\psi_\nu^{(\eta)}$ is the desired nonzero component. ∎

3.2. LEMMA. *Suppose that $\nu, \nu' \in \mathbf{N}_0^q$ with $|\nu'|, |\nu| \geq n$. Furthermore, suppose that there exists $\eta \in \mathbf{N}_0^q(n)$ with*

$$\psi_\nu^{(\eta)} = \psi_{\nu'}^{(\eta)} \neq 0.$$

Then $\nu = \nu'$.

PROOF. By Corollary 2.6, $\psi_\nu^{(\eta)}(\zeta) = C_\nu \zeta^{\nu-\eta}$ and $\psi_{\nu'}^{(\eta)}(\zeta) = C_{\nu'} \zeta^{\nu'-\eta}$, where $C_\nu, C_{\nu'}$ are nonzero constants. Hence $\nu = \nu'$. ∎

Lemmas 3.1 and 3.2 show that $f_{\nu,n}$ can be recovered from a single nonzero summand $\psi_\nu^{(\eta)}(\zeta)\bar{v}^\eta/\eta!$ of $\phi_\nu(\zeta, v)$ in (3.1). However, there may be more than one nonzero summand of ϕ_ν. Our approach to the inversion problem is to first develop a method for isolating a nonzero summand of ϕ_ν.

We will construct a projection P that when applied to ϕ_ν returns the η_ν-th summand $\psi_\nu^{(\eta_\nu)}\bar{v}^{\eta_\nu}/\eta_\nu!$ in (3.1), where $\eta_\nu \in \mathbf{N}_0^q(n)$ is maximal in the lexicographic order $\prec$ with respect to $\psi_\nu^{(\eta_\nu)}$ being nonzero. Restricting the ω_n action

of Definition 1.10 to K, we see that the K action on $\bar{\mathcal{P}}(n, \mathbf{C}^q)$ (thought of as the fiber over $\zeta = 0$ of the holomorphic, homogeneous vector bundle $\mathbf{E}_n$ for G over G/K) is simply the natural action of $U(q)$ by left translation. Notice that each $\bar{v}^\eta$ is a weight vector for this representation, and that the lexicographic order $\prec$ corresponds to the order on these weight vectors, from lowest to highest. Hence P will return the summand of ϕ_ν of highest possible weight with respect to this action of K on $\bar{\mathcal{P}}(n, \mathbf{C}^q)$.

3.3. DEFINITION. For each $j \in \{1, 2, \ldots, q\}$ define $Q_j : \mathcal{O}(\mathbf{B}^q, \bar{\mathcal{P}}(n, \mathbf{C}^q)) \to \mathcal{O}(\mathbf{B}^q, \bar{\mathcal{P}}(n, \mathbf{C}^q))$ by

$$Q_j f(\zeta, v) = f(\zeta_1, \ldots, \zeta_{j-1}, 0, \zeta_{j+1}, \ldots, \zeta_q, v).$$

3.4. DEFINITION. If $\eta \in \mathbf{N}_0^q(n)$ has an immediate successor $\tilde{\eta}$ with respect to the partial order $\prec$, let $s(\eta)$ be the index at which η first disagrees with $\tilde{\eta}$. If η has no immediate successor, i.e., $\eta = (n, 0, \ldots, 0)$, let $s(\eta) = 0$.

For example, $\eta = (3, 1, 2, 4, 0)$ has immediate successor $\tilde{\eta} = (3, 1, 3, 3, 0)$. So $s(\eta) = 3$, as η and $\tilde{\eta}$ first disagree in the third component.

3.5. DEFINITION. Suppose that $\eta \in \mathbf{N}_0^q(n)$. Define $P_\eta : \mathcal{O}(\mathbf{B}^q, \bar{\mathcal{P}}(n, \mathbf{C}^q)) \to \mathcal{O}(\mathbf{B}^q, \bar{\mathcal{P}}(n, \mathbf{C}^q))$ by

$$P_\eta := \prod_{j=1}^{s(\eta)} Q_j$$

where $s(\eta)$ is as in Definition 3.4. If $\eta = (n, 0, \ldots, 0)$, then $P_\eta := Id$.

3.6. DEFINITION. Define $P : \mathcal{O}(\mathbf{B}^q, \bar{\mathcal{P}}(n, \mathbf{C}^q)) \to \mathcal{O}(\mathbf{B}^q, \bar{\mathcal{P}}(n, \mathbf{C}^q))$ by

$$(P\phi)(\zeta, v) = \sum_{\eta \in \mathbf{N}_0^q(n)} P_\eta \frac{\bar{v}^\eta}{\eta!} \frac{\partial^n \phi}{\partial \bar{v}^\eta} (\zeta, v).$$

The role of the operator $(\bar{v}^\eta/\eta!)(\partial^n/\partial \bar{v}^\eta)$ in Definition 3.6 is to single out the terms containing $\bar{v}^\eta$ in ϕ, as in (2.1). Then P_η is applied only to those terms.

For example, let $q = 3$ and $n = 2$. In this case the following table gives possible values of $\eta \in \mathbf{N}_0^3(2)$, in increasing lexicographic order, and the corresponding mappings P_η.

η	$s(\eta)$	P_η
$(0, 0, 2)$	2	$Q_1 Q_2$
$(0, 1, 1)$	2	$Q_1 Q_2$
$(0, 2, 0)$	1	Q_1
$(1, 0, 1)$	2	$Q_1 Q_2$
$(1, 1, 0)$	1	Q_1
$(2, 0, 0)$	0	Id

Any function $\phi \in \mathcal{O}(\mathbf{B}^3, \bar{\mathcal{P}}(2, \mathbf{C}^3))$ is written using (2.1) as

$$\phi(\zeta, v) = \psi^{(0,0,2)}(\zeta)\frac{\bar{v}_3^2}{2!} + \psi^{(0,1,1)}(\zeta)\bar{v}_2\bar{v}_3 + \cdots + \psi^{(2,0,0)}(\zeta)\frac{\bar{v}_1^2}{2!},$$

and the map P acts by P_η, as given in the table, on the term containing $\bar{v}^\eta$.

Now let $f \in \mathcal{F}_2^{1,3}$ be given by $f(z) = z_1^3\bar{z}_2\bar{z}_3^3\bar{z}_4$, so that $\nu = (1,3,1)$, in the notation of Corollary 2.6. The $\eta \in \mathbf{N}_0^3(2)$ for which $\nu \geq \eta$ are $(0,1,1)$, $(0,2,0)$, $(1,0,1)$, and $(1,1,0)$. It follows from Corollary 2.6 that

$$(\Phi_2 f)(\zeta, v) = 18\zeta_1\zeta_2^2\bar{v}_2\bar{v}_3 + 36\zeta_1\zeta_2\zeta_3\frac{\bar{v}_2^2}{2!} + 6\zeta_2^3\bar{v}_1\bar{v}_3 + 18\zeta_2^2\zeta_3\bar{v}_1\bar{v}_2.$$

Using Definition 3.6 and the table above, we compute that for $\phi = \Phi_2 f$,

$$P\phi(\zeta, v) = 18\zeta_2^2\zeta_3\bar{v}_1\bar{v}_2.$$

Notice that only one term of $\phi(\zeta, v)$ remains, the term indexed by the maximal $\eta \in \mathbf{N}_0^3(2)$ with respect to lexicographic order for which $\psi^{(\eta)} \neq 0$. This is no coincidence, as is shown by the next lemma.

3.7. LEMMA. *Fix $\nu \in \mathbf{N}_0^q$ with $|\nu| \geq n$. Let $f_{\nu,n}(z) = z_1^{|\nu|-n}\bar{z}_S^\nu$, and let $\Phi_n f_{\nu,n} = \phi_\nu = \sum_{\eta\in\mathbf{N}_0^q(n)} \psi_\nu^{(\eta)}\bar{v}^\eta/\eta!$, as in (3.1). Let $\alpha \in \mathbf{N}_0^q(n)$ be maximal in the order $\prec$ with respect to $\psi_\nu^{(\alpha)}$ being nonzero. Then $P_\alpha\psi_\nu^{(\alpha)}\bar{v}^\alpha/\alpha! = \psi_\nu^{(\alpha)}\bar{v}^\alpha/\alpha!$ and for $\eta \prec \alpha$, $P_\eta\psi_\nu^{(\eta)}\bar{v}^\eta/\eta! = 0$.*

PROOF. Since $\psi_\nu^{(\eta)}$ is a monomial in ζ, then either $P_\eta\psi_\nu^{(\eta)}\bar{v}^\eta/\eta! = \psi_\nu^{(\eta)}\bar{v}^\eta/\eta!$ or $P_\eta\psi_\nu^{(\eta)}\bar{v}^\eta/\eta! = 0$ for any $\eta \in \mathbf{N}_0^q(n)$. We first show that $P_\alpha\psi_\nu^{(\alpha)}\bar{v}^\alpha/\alpha! = \psi_\nu^{(\alpha)}\bar{v}^\alpha/\alpha!$.

Suppose that $\alpha = (n, 0, \ldots, 0)$. Then by Definition 3.5, $P_\alpha = \mathrm{Id}$, so we have $P_\alpha\psi_\nu^{(\alpha)}\bar{v}^\alpha/\alpha! = \psi_\nu^{(\alpha)}\bar{v}^\alpha/\alpha!$ trivially.

So we now assume that $\alpha = (\alpha_1, \ldots, \alpha_q) \neq (n, 0, \ldots, 0)$. Let $\tilde{\alpha}$ be the immediate successor of α within $\mathbf{N}_0^q(n)$. Note that $\tilde{\alpha}_i = \alpha_i$ for $1 \leq i < s(\alpha)$. Also, $\tilde{\alpha}_{s(\alpha)} > \alpha_{s(\alpha)}$, and $|\tilde{\alpha}| = |\alpha| = n$. It follows that there must be a nonzero component α_t of α with $s(\alpha) < t \leq q$. Also, by Corollary 2.6, we have that

$$\psi_\nu^{(\alpha)}(\zeta) = C_\alpha\zeta_1^{\nu_1-\alpha_1}\cdots\zeta_q^{\nu_q-\alpha_q},$$

where $\nu \geq \alpha$ and C_α is a constant. For a contradiction, assume that $P_\alpha\psi_\nu^{(\alpha)}\bar{v}^\alpha/\alpha! = 0$. Then, by Definition 3.5, there exists $k \in \{1, \ldots, s(\alpha)\}$ with $\nu_k - \alpha_k > 0$. Consider $\alpha' = \alpha - \epsilon_t + \epsilon_k$. We have that $\alpha' \in \mathbf{N}_0^q(n)$ with $\alpha' \succ \alpha$. Furthermore, by Corollary 2.6, $\psi_\nu^{(\alpha')} \neq 0$. This contradicts the maximality of α.

Fix $\eta \prec \alpha$. The proof will be concluded by showing that $P_\eta\psi_\nu^{(\eta)}\bar{v}^\eta/\eta! = 0$. If $\psi_\nu^{(\eta)} = 0$ then the result is immediate. So, for a contradiction, assume that $\psi_\nu^{(\eta)} \neq 0$ and that $P_\eta\psi_\nu^{(\eta)}\bar{v}^\eta/\eta! = \psi_\nu^{(\eta)}\bar{v}^\eta/\eta!$. Definition 3.5 and Corollary 2.6 then imply that $\nu_i - \eta_i = 0$ for $i \in \{1, \ldots, s(\eta)\}$. Now, since $\alpha \succ \eta$, there exists $k \in \{1, \ldots, s(\eta)\}$ such that $\alpha_k > \eta_k$. However, then $\nu_k - \alpha_k < 0$, and so by Corollary 2.6, $\psi_\nu^{(\alpha)} = 0$. This contradicts the fact that $\psi_\nu^{(\alpha)} \neq 0$. ∎

We now construct another mapping of $\mathcal{O}(\mathbf{B}^q, \bar{\mathcal{P}}(n, \mathbf{C}^q))$ into itself that we will use to weight the mapping P. Essentially, we will be assigning to each element $\phi \in \mathcal{O}(\mathbf{B}^q, \bar{\mathcal{P}}(n, \mathbf{C}^q))$ a solution of a certain partial differential equation.

3.8. DEFINITION. Suppose $f \in \mathcal{O}(\mathbf{B}^q, \bar{\mathcal{P}}(n, \mathbf{C}^q))$, $\zeta \in \mathbf{B}^q$, and $v \in \mathbf{C}^q$. For a given non-negative integer m and an index $j \in \{1, \ldots, q\}$, we define the mapping $I(m, j) : \mathcal{O}(\mathbf{B}^q, \bar{\mathcal{P}}(n, \mathbf{C}^q)) \to \mathcal{O}(\mathbf{B}^q, \bar{\mathcal{P}}(n, \mathbf{C}^q))$ by

$$(I(m,j)f)(\zeta, v) := \int_0^1 \int_0^{t_m} \cdots \int_0^{t_2} f(\zeta_1, \ldots, \zeta_{j-1}, t_1\zeta_j, \zeta_{j+1}, \ldots, \zeta_q, v)\, dt_1 dt_2 \ldots dt_m,$$

with the convention that $(I(0,j)f)(\zeta, v) = f(\zeta, v)$.

For example, if $f(\zeta, v) = \zeta_j^k \bar{v}_1^n$, then $(I(m,j)f)(\zeta, v) = \dfrac{k!}{(k+m)!}\zeta_j^k \bar{v}_1^n$.

3.9. DEFINITION. Define $F_\eta : \mathcal{O}(\mathbf{B}^q, \bar{\mathcal{P}}(n, \mathbf{C}^q)) \to \mathcal{O}(\mathbf{B}^q, \bar{\mathcal{P}}(n, \mathbf{C}^q))$ by

$$(F_\eta f)(\zeta, v) := \Big(I(\eta_1, 1)I(\eta_2, 2)\ldots I(\eta_q, q)f\Big)(\zeta, v),$$

where $\eta \in \mathbf{N}_0^q(n)$.

Suppose that $f \in \mathcal{O}(\mathbf{B}^q, \bar{\mathcal{P}}(n, \mathbf{C}^q))$ and

$$f(\zeta, v) = \sum_{\eta \in \mathbf{N}_0^q(n)} \sum_{\gamma \in \mathbf{N}_0^q} a_{\eta,\gamma} \zeta^\gamma \bar{v}^\eta$$

is the Taylor series expansion for f centered at the origin. Since the series above converges uniformly on compact subsets of $\mathbf{B}^q$, it follows from Definition 3.8 that if $\rho \in \mathbf{N}_0^q(n)$,

$$\begin{aligned} F_\rho f(\zeta, v) &= \sum_{\eta \in \mathbf{N}_0^q(n)} \sum_{\gamma \in \mathbf{N}_0^q} a_{\eta,\gamma} F_\rho(\zeta^\gamma \bar{v}^\eta) \\ &= \sum_{\eta \in \mathbf{N}_0^q(n)} \sum_{\gamma \in \mathbf{N}_0^q} \frac{a_{\eta,\gamma}\gamma!}{(\gamma+\rho)!} \zeta^\gamma \bar{v}^\eta. \end{aligned} \tag{3.2}$$

Now, we weight the mapping P by the F_η to obtain a new mapping that will play a large role in the rest of the paper.

3.10. DEFINITION. Define $L : \mathcal{O}(\mathbf{B}^q, \bar{\mathcal{P}}(n, \mathbf{C}^q)) \to \mathcal{O}(\mathbf{B}^q, \bar{\mathcal{P}}(n, \mathbf{C}^q))$ by

$$(L\phi)(\zeta, v) = \sum_{\eta \in \mathbf{N}_0^q(n)} F_\eta P_\eta \bar{v}^\eta \frac{\partial^n \phi}{\partial \bar{v}^\eta}(\zeta, v).$$

For example, recall that if $f(z) = z_1^3 \bar{z}_2 \bar{z}_3^3 \bar{z}_4$, and $\phi = \Phi_2 f$, then

$$P\phi(\zeta, v) = 18\zeta_2^2 \zeta_3 \bar{v}_1 \bar{v}_2.$$

Applying (3.2) and Definition 3.10, we compute that

$$L\phi(\zeta, v) = 18\frac{(0,2,1)!}{((0,2,1)+(1,1,0))!}\zeta_2^2\zeta_3\bar{v}_1\bar{v}_2 = 6\zeta_2^2\zeta_3\bar{v}_1\bar{v}_2.$$

For $\phi \in \mathcal{O}(\mathbf{B}^q, \bar{\mathcal{P}}(n, \mathbf{C}^q))$, we express ϕ as the sum

$$\phi(\zeta, v) = \sum_{\eta \in \mathbf{N}_0^q(n)} \psi^{(\eta)}(\zeta)\frac{\bar{v}^\eta}{\eta!}$$

as in (2.1). For a compact subset A of $\mathbf{B}^q$, set

$$\|\phi\|_A^2 := \sum_{\eta \in \mathbf{N}_0^q(n)} |\psi^{(\eta)}|_A^2, \tag{3.3}$$

where $|\cdot|_A$ is the supremum norm over A. We endow $\mathcal{O}(\mathbf{B}^q, \bar{\mathcal{P}}(n, \mathbf{C}^q))$ with the topology of uniform convergence on compact subsets of $\mathbf{B}^q$, i.e., the topology induced by the norm of (3.3).

3.11. LEMMA. *The mapping* $L : \mathcal{O}(\mathbf{B}^q, \bar{\mathcal{P}}(n, \mathbf{C}^q)) \to \mathcal{O}(\mathbf{B}^q, \bar{\mathcal{P}}(n, \mathbf{C}^q))$ *is continuous.*

PROOF. The remarks after Definition 3.9 serve to show that F_η is continuous for each $\eta \in \mathbf{N}_0^q(n)$. It remains to show that P_η is continuous.

Let $\phi \in \mathcal{O}(\mathbf{B}^q, \bar{\mathcal{P}}(n, \mathbf{C}^q))$. Suppose that $\{\phi_k\}$ is a sequence of functions in $\mathcal{O}(\mathbf{B}^q, \bar{\mathcal{P}}(n, \mathbf{C}^q))$ such that $\{\phi_k\}$ converges to ϕ uniformly on compact subsets of $\mathbf{B}^q$, in the norm of (3.3). Let A be compact in $\mathbf{B}^q$ and for $1 \le j \le q$, let A_j be a compact subset of $\mathbf{B}^q$ containing the natural projection of A onto the set $\{\zeta \in \mathbf{B}^q \mid \zeta_j = 0\}$. Then

$$\|Q_j\phi - Q_j\phi_k\|_A \le \|\phi - \phi_k\|_{A_j}.$$

It follows that Q_j is continuous and hence P_η is continuous. ∎

We are now ready to invert the transform Φ_n of section 2.

3.12. THEOREM. *Suppose that* $\phi \in \Phi_n(\mathcal{F}_n^{1,q})$, *and that* $\phi = \Phi_n f$, *for* $f \in \mathcal{F}_n^{1,q}$. *Then, for* $z \in \mathbf{C}^{q+1}$,

$$f(z) = \lim_{t \to 1^-} \frac{d^q}{dt^q}\left[t^q \int_{\mathbf{B}^q}\int_{\mathbf{C}^q} (L\phi)(t\zeta, v)e^{(z_1\bar{\zeta}+v^T)\bar{z}_S}\, e^{-|v|^2}\, dm(v)dm(\zeta)\right].$$

PROOF. The theorem is proven in two steps. First, given $\nu \in \mathbf{N}_0^q$ with $|\nu| \ge n$, we let $f_{\nu,n}(z) = z_1^{|\nu|-n}\bar{z}_S^\nu$ and we let $\phi_\nu = \Phi_n f_{\nu,n}$. We will prove that the theorem holds for ϕ_ν. Then we will show that the theorem holds for an arbitrary $\phi \in \Phi_n(\mathcal{F}_n^{1,q})$.

So, let ν be as above, and let $\eta_\nu \in \mathbf{N}_0^q(n)$ be such that $P_{\eta_\nu}(\psi_\nu^{(\eta_\nu)}\bar{v}^{\eta_\nu}/\eta_\nu!) \neq 0$. Then by Lemma 3.7 and Definition 3.10,

$$\begin{aligned}(L\phi_\nu)(\zeta,v) &= \sum_{\eta\in\mathbf{N}_0^q(n)} F_\eta P_\eta \bar{v}^\eta \frac{\partial^n \phi_\nu}{\partial \bar{v}^\eta}(\zeta,v) \\ &= F_{\eta_\nu} P_{\eta_\nu} \frac{\nu!(|\nu|-n)!}{(\nu-\eta_\nu)!}\zeta^{\nu-\eta_\nu}\bar{v}^{\eta_\nu} \\ &= (|\nu|-n)!\zeta^{\nu-\eta_\nu}\bar{v}^{\eta_\nu}. \qquad (3.4)\end{aligned}$$

Hence, by using (3.4), (1.3), and Lemma 1.9, we compute that

$$\begin{aligned}&t^q \int_{\mathbf{B}^q}\int_{\mathbf{C}^q} (L\phi_\nu)(t\zeta,v)e^{(z_1\bar{\zeta}+v^T)\bar{z}_S}\, e^{-|v|^2}\, dm(v)dm(\zeta) \\ &\quad = t^{|\nu|-n+q}(|\nu|-n)! \int_{\mathbf{B}^q}\int_{\mathbf{C}^q} \zeta^{\nu-\eta_\nu}\bar{v}^{\eta_\nu} e^{(z_1\bar{\zeta}+v^T)\bar{z}_S}\, e^{-|v|^2} dm(v)dm(\zeta) \\ &\quad = t^{|\nu|-n+q}\bar{z}_S^{\eta_\nu}(|\nu|-n)! \int_{\mathbf{B}^q} \zeta^{\nu-\eta_\nu} \exp\left[z_1\left(\sum_{j=1}^q \overline{z_{j+1}\zeta_j}\right)\right] dm(\zeta) \\ &\quad = t^{|\nu|-n+q}\left[\frac{z_1^{|\nu|-n}\bar{z}_S^\nu(|\nu|-n)!}{(\nu-\eta_\nu)!}\right] \int_{\mathbf{B}^q} \zeta^{\nu-\eta_\nu}\bar{\zeta}^{\nu-\eta_\nu} dm(\zeta) \\ &\quad = t^{|\nu|-n+q}\left[\frac{z_1^{|\nu|-n}\bar{z}_S^\nu(|\nu|-n)!}{(|\nu|-n+q)!}\right],\end{aligned}$$

and so

$$\begin{aligned}&\lim_{t\to 1^-}\frac{d^q}{dt^q}\left[t^q \int_{\mathbf{B}^q}\int_{\mathbf{C}^q} (L\phi_\nu)(t\zeta,v)e^{(z_1\bar{\zeta}+v^T)\bar{z}_S}\, e^{-|v|^2}\, dm(v)dm(\zeta)\right] \\ &\qquad = \lim_{t\to 1^-}\left(\frac{d^q}{dt^q} t^{|\nu|-n+q}\left[\frac{z_1^{|\nu|-n}\bar{z}_S^\nu(|\nu|-n)!}{(|\nu|-n+q)!}\right]\right) \\ &\qquad = f_{\nu,n}(z).\end{aligned}$$

This completes the first part of the proof.

Now let ϕ be an arbitrary element of $\Phi_n(\mathcal{F}_n^{1,q})$, $\phi = \Phi_n f$ for $f \in \mathcal{F}_n^{1,q}$. Put

$$f(z) = \sum_{\substack{\nu\in\mathbf{N}_0^q \\ |\nu|\geq n}} a_\nu f_{\nu,n}(z).$$

This series converges uniformly on compact subsets of $\mathbf{C}^{q+1}$ *and* in the norm on $\mathcal{F}_n^{1,q}$ induced by $\langle\cdot,\cdot\rangle$. Hence by Lemma 2.3, given $\zeta \in \mathbf{B}^q$,

$$\phi(\zeta,v) = \sum_{\substack{\nu\in\mathbf{N}_0^q \\ |\nu|\geq n}} a_\nu \phi_\nu(\zeta,v). \qquad (3.5)$$

Moreover, the series in (3.5) constitutes a Taylor series expansion for ϕ, and so the series converges uniformly on compact subsets of $\mathbf{B}^q$. Hence by (1.3) and

Lemma 3.11, for $t \in [0,1)$ we have

$$\int_{\mathbf{B}^q}\int_{\mathbf{C}^q} L\Big(\sum_{\substack{\nu\in\mathbf{N}_0^q\\ |\nu|\geq n}} a_\nu\phi_\nu\Big)(t\zeta,v)e^{(z_1\bar{\zeta}+v^T)\bar{z}_S}\, e^{-|v|^2}\, dm(v)dm(\zeta)$$

$$= \int_{\mathbf{B}^q}\int_{\mathbf{C}^q} \sum_{\substack{\nu\in\mathbf{N}_0^q\\ |\nu|\geq n}} a_\nu(L\phi_\nu)(t\zeta,v)e^{(z_1\bar{\zeta}+v^T)\bar{z}_S}\, e^{-|v|^2}\, dm(v)dm(\zeta)$$

$$= \int_{\mathbf{B}^q} \sum_{\substack{\nu\in\mathbf{N}_0^q\\ |\nu|\geq n}} t^{|\nu|-n}a_\nu(|\nu|-n)!\zeta^{\nu-\eta_\nu}\bar{z}_S^{\eta_\nu}\exp\Big[z_1\Big(\sum_{j=1}^{q}\overline{z_{j+1}\zeta_j}\Big)\Big]\, dm(\zeta). \tag{3.6}$$

Notice that we may exchange integration over $\mathbf{C}^q$ with summation in the equation preceding (3.6) due to the absolute convergence of the integral in v. We finish by showing that we may interchange integration and summation in the right hand side of (3.6). The theorem will then follow immediately from the first part of the proof. Thus by the Fubini-Tonelli theorem, it suffices to show that

$$\sum_{\substack{\nu\in\mathbf{N}_0^q\\ |\nu|\geq n}} |a_\nu|(|\nu|-n)!t^{|\nu|-n}\int_{\mathbf{B}^q}\Big|\zeta^{\nu-\eta_\nu}\bar{z}_S^{\eta_\nu}\exp\Big[z_1\Big(\sum_{j=1}^{q}\overline{z_{j+1}\zeta_j}\Big)\Big]\Big|\, dm(\zeta) < \infty.$$

Note that for a fixed $z \in \mathbf{C}^{q+1}$, the function

$$\zeta \mapsto \bar{z}_S^{\eta_\nu}\exp\Big[z_1\Big(\sum_{j=1}^{q}\overline{z_{j+1}\zeta_j}\Big)\Big]$$

is bounded in modulus on $\mathbf{B}^q$. Hence there is a positive constant c_z such that

$$\sum_{\substack{\nu\in\mathbf{N}_0^q\\ |\nu|\geq n}} |a_\nu|(|\nu|-n)!t^{|\nu|-n}\int_{\mathbf{B}^q}\Big|\zeta^{\nu-\eta_\nu}\bar{z}_S^{\eta_\nu}\exp\Big[z_1\Big(\sum_{j=1}^{q}\overline{z_{j+1}\zeta_j}\Big)\Big]\Big| dm(\zeta)$$

$$\leq c_z \sum_{\substack{\nu\in\mathbf{N}_0^q\\ |\nu|\geq n}} |a_\nu|(|\nu|-n)!t^{|\nu|-n}\int_{\mathbf{B}^q}|\zeta^{\nu-\eta_\nu}|dm(\zeta). \tag{3.7}$$

Now by applying the Cauchy-Schwarz inequality to the integral in (3.7) and

using Lemma 1.9 we compute that

$$c_z \sum_{\substack{\nu\in\mathbf{N}_0^q \\ |\nu|\geq n}} |a_\nu|(|\nu|-n)!t^{|\nu|-n}\int_{\mathbf{B}^q}|\zeta^{\nu-\eta_\nu}|dm(\zeta)$$

$$\leq c_z' \sum_{\substack{\nu\in\mathbf{N}_0^q \\ |\nu|\geq n}} |a_\nu|(|\nu|-n)!t^{|\nu|-n}\Big(\frac{(\nu-\eta_\nu)!}{(|\nu|-n+q)!}\Big)^{\frac{1}{2}}$$

$$\leq c_z' \sum_{\substack{\nu\in\mathbf{N}_0^q \\ |\nu|\geq n}} |a_\nu|((|\nu|-n)!)^{\frac{1}{2}}(\nu!)^{\frac{1}{2}}\Big(\frac{t^{2(|\nu|-n)}}{(|\nu|-n+q)\dots(|\nu|-n+1)}\Big)^{\frac{1}{2}}$$

(3.8)

$$\leq c_z'\Big[\sum_{\substack{\nu\in\mathbf{N}_0^q \\ |\nu|\geq n}} |a_\nu|^2(|\nu|-n)!\nu!\Big]^{\frac{1}{2}}\Big[\sum_{\substack{\nu\in\mathbf{N}_0^q \\ |\nu|\geq n}} \frac{t^{2(|\nu|-n)}}{(|\nu|-n+q)\dots(|\nu|-n+1)}\Big]^{\frac{1}{2}}.$$

By (1.2), the first factor of (3.8) is simply $\langle f,f\rangle^{\frac{1}{2}}$, while the second factor is finite by Lemma 1.4. Hence the product in (3.8) is finite, and so the proof is concluded. ■

4. A Hilbert Space Structure for $\Phi_n(\mathcal{F}_n^{1,q})$

Recalling Theorems 1.7 and 2.2, observe that for $n\geq 0$ the space $\Phi_n(\mathcal{F}_n^{1,q})$ is an irreducible subspace of $\mathcal{O}(\mathbf{B}^q,\bar{\mathcal{P}}(n,\mathbf{C}^q))$ under the representation ω_n of $U(1,q)$. Let ϑ_n denote this subrepresentation. One would like to explicitly describe a Hilbert space structure on $\Phi_n(\mathcal{F}_n^{1,q})$ with the property that ϑ_n and σ_n will be unitarily equivalent via Φ_n. It is worth pointing out that σ_n is not in the discrete series for any $n\geq 0$ if $p=1$ and $q>1$. If $p=q=1$ then the ladder representation of spin $n\geq 1$ is in the discrete series and the representation with spin $n=0$ is a limit of discrete series (see Theorem 4.4 of [**3**]).

4.1. DEFINITION. For $\phi_1,\phi_2\in\mathcal{O}(\mathbf{B}^q,\bar{\mathcal{P}}(n,\mathbf{C}^q))$, define

$$((\phi_1,\phi_2))_{n,q} := \lim_{t\to 1^-}\frac{d^q}{dt^q}\Big[t^q\int_{\mathbf{B}^q}\int_{\mathbf{C}^q}(L\phi_1)(t\zeta,v)\overline{\phi_2(\zeta,v)}\,e^{-|v|^2}\,dm(v)dm(\zeta)\Big]$$

$$= \lim_{t\to 1^-}\frac{d^q}{dt^q}\Big[t^q\int_{\mathbf{B}^q}\langle(L\phi_1)(t\zeta,\cdot),\phi_2(\zeta,\cdot)\rangle\,dm(\zeta)\Big],$$

whenever the integral above converges. Here L is as in Definition 3.10.

For $\phi\in\mathcal{O}(\mathbf{B}^q,\bar{\mathcal{P}}(n,\mathbf{C}^q))$, put $\|\phi\|_{n,q}^2 := ((\phi,\phi))_{n,q}$. It is easy to show that on the set $\{\phi\in\mathcal{O}(\mathbf{B}^q,\bar{\mathcal{P}}(n,\mathbf{C}^q)) \mid \|\phi\|_{n,q}^2<\infty\}$, the form $((\cdot,\cdot))_{n,q}$ is hermitian and positive semi-definite. We next examine its behavior on $\Phi_n(\mathcal{F}_n^{1,q})$.

4.2. THEOREM. *Let $\phi_1,\phi_2\in\Phi_n(\mathcal{F}_n^{1,q})$ with $\phi_1=\Phi_n f_1$ and $\phi_2=\Phi_n f_2$ for $f_1,f_2\in\mathcal{F}_n^{1,q}$. Then*

$$((\phi_1,\phi_2))_{n,q}=\langle f_1,f_2\rangle.$$

Hence $((\cdot,\cdot))_{n,q}$ *defines a positive definite hermitian form on* $\Phi_n(\mathcal{F}_n^{1,q})$.

PROOF. Put

$$f_1(z) = \sum_{\substack{\nu\in\mathbf{N}_0^q \\ |\nu|\geq n}} a_\nu f_{\nu,n}(z) \quad \text{and} \quad f_2(z) = \sum_{\substack{\nu\in\mathbf{N}_0^q \\ |\nu|\geq n}} b_\nu f_{\nu,n}(z),$$

where $f_{\nu,n}$ is as in Corollary 2.6. Correspondingly, according to Lemma 2.3, we have

$$\phi_1(\zeta,v) = \sum_{\substack{\nu\in\mathbf{N}_0^q \\ |\nu|\geq n}} a_\nu \phi_\nu(\zeta,v) \quad \text{and} \quad \phi_2(\zeta,v) = \sum_{\substack{\nu\in\mathbf{N}_0^q \\ |\nu|\geq n}} b_\nu \phi_\nu(\zeta,v),$$

where $\phi_\nu = \Phi_n f_{\nu,n}$. Then by using the series expansions above, along with Lemma 3.11, Corollary 2.6, (3.4), and Lemma 1.2, for $t \in [0,1)$,

$$\begin{aligned}
&\int_{\mathbf{B}^q}\int_{\mathbf{C}^q} (L\phi_1)(t\zeta,v)\overline{\phi_2(\zeta,v)}e^{-|v|^2}\,dm(v)dm(\zeta) \\
&\quad = \int_{\mathbf{B}^q}\int_{\mathbf{C}^q} \sum_{\substack{\nu\in\mathbf{N}_0^q \\ |\nu|\geq n}} \sum_{\substack{\tilde\nu\in\mathbf{N}_0^q \\ |\tilde\nu|\geq n}} a_\nu \bar b_{\tilde\nu} (L\phi_\nu)(t\zeta,v)\overline{\phi_{\tilde\nu}(\zeta,v)}\, e^{-|v|^2}\,dm(v)dm(\zeta) \\
&\quad = \int_{\mathbf{B}^q}\Bigg[\sum_{\substack{\nu,\tilde\nu\in\mathbf{N}_0^q \\ |\nu|,|\tilde\nu|\geq n \\ \tilde\nu\geq\eta_\nu}} a_\nu \bar b_{\tilde\nu} t^{|\nu|-n} \frac{(|\nu|-n)!(|\tilde\nu|-n)!\tilde\nu!}{(\tilde\nu-\eta_\nu)!}\zeta^{\nu-\eta_\nu}\bar\zeta^{\tilde\nu-\eta_\nu}\Bigg] dm(\zeta). \qquad (4.1)
\end{aligned}$$

Since $t \in [0,1)$, by the dominated convergence theorem we may interchange integration and summation in (4.1). After doing so and applying Lemma 1.9, the only nonzero terms remaining in the sum are those for which $\nu = \tilde\nu$, yielding that

$$\begin{aligned}
&\int_{\mathbf{B}^q}\Bigg[\sum_{\substack{\nu,\tilde\nu\in\mathbf{N}_0^q \\ |\nu|,|\tilde\nu|\geq n \\ \tilde\nu\geq\eta_\nu}} a_\nu \bar b_{\tilde\nu} t^{|\nu|-n} \frac{(|\nu|-n)!(|\tilde\nu|-n)!\tilde\nu!}{(\tilde\nu-\eta_\nu)!}\zeta^{\nu-\eta_\nu}\bar\zeta^{\tilde\nu-\eta_\nu}\Bigg] dm(\zeta) \\
&\qquad = \sum_{\substack{\nu\in\mathbf{N}_0^q \\ |\nu|\geq n}} a_\nu \bar b_\nu t^{|\nu|-n}[(|\nu|-n)!]^2 \frac{\nu!}{(|\nu|-n+q)!} \\
&\qquad = \sum_{\substack{\nu\in\mathbf{N}_0^q \\ |\nu|\geq n}} \frac{a_\nu \bar b_\nu t^{|\nu|-n}\nu!(|\nu|-n)!}{(|\nu|-n+q)(|\nu|-n+(q-1))\dots(|\nu|-n+1)}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\lim_{t\to 1^-}\frac{d^q}{dt^q}\Bigg[t^q\int_{\mathbf{B}^q}\int_{\mathbf{C}^q}(L\phi_1)(t\zeta,v)\overline{\phi_2(\zeta,v)}\,e^{-|v|^2}\,dm(v)dm(\zeta)\Bigg] \\
&\qquad = \sum_{\substack{\nu\in\mathbf{N}_0^q \\ |\nu|\geq n}} a_\nu\bar b_\nu(|\nu|-n)!\nu! \\
&\qquad = \langle f_1, f_2\rangle. \quad \blacksquare
\end{aligned}$$

4.3. COROLLARY. *The vector space $\Phi_n(\mathcal{F}_n^{1,q})$, when endowed with the inner product $(\!(\cdot,\cdot)\!)_{n,q}$, is a Hilbert space.*

PROOF. By Theorem 4.2, $\Phi_n(\mathcal{F}_n^{1,q})$ is a pre-Hilbert space. So all we need to show is that $\Phi_n(\mathcal{F}_n^{1,q})$ is complete. Suppose that $\{\phi_k\}$ is a sequence in $\Phi_n(\mathcal{F}_n^{1,q})$ that is Cauchy in the norm induced by $(\!(\cdot,\cdot)\!)_{n,q}$, and suppose that $\phi_k = \Phi_n(f_k)$ for $f_k \in \mathcal{F}_n^{1,q}$. Then by Theorem 4.2, the sequence $\{f_k\}$ is Cauchy in the Hilbert Space $\mathcal{F}_n^{1,q}$. Hence there is a function $f \in \mathcal{F}_n^{1,q}$ such that f_k converges to f in norm. Let $\phi = \Phi_n f$. Then $\phi \in \Phi_n(\mathcal{F}_n^{1,q})$ and ϕ_k converges to ϕ in norm. ∎

4.4. COROLLARY. *The representation ϑ_n is irreducible and unitary on the Hilbert space $\Phi_n(\mathcal{F}_n^{1,q})$ and is unitarily equivalent to σ_n via Φ_n.*

PROOF. This follows from Theorems 2.2 and 4.2. ∎

At this point one may wonder whether the inner product $(\!(\cdot,\cdot)\!)_{n,q}$ characterizes $\Phi_n(\mathcal{F}_n^{1,q})$. In general, this inner product is not enough to characterize $\Phi_n(\mathcal{F}_n^{1,q})$. However, when we put the inner product together with the differential equations of Theorem 2.4, we are able to combine our results with those of [**4**] to obtain a complete characterization of $\Phi_n(\mathcal{F}_n^{1,q})$.

For $q > 1$, let $\mathcal{D}_{1,q}$ denote the set of differential operators consisting of all 2×2-minors of the matrix of differential operators

$$\Box_{1,q} = \begin{pmatrix} \frac{\partial}{\partial \zeta_1} & \cdots & \frac{\partial}{\partial \zeta_q} \\ \frac{\partial}{\partial \bar{v}_1} & \cdots & \frac{\partial}{\partial \bar{v}_q} \end{pmatrix}$$

of Theorem 2.4. If $q = 1$, we will say that $\mathcal{D}_{1,1} = \emptyset$. Also, let

$$\mathcal{X}_n^{1,q} = \{\phi \in \mathcal{O}(\mathbf{B}^q, \bar{\mathcal{P}}(n, \mathbf{C}^q)) \mid \|\phi\|_{n,q}^2 < \infty \text{ and } D\phi = 0 \text{ for all } D \in \mathcal{D}_{1,q}\}.$$

4.5. THEOREM. *Let $n \in \mathbf{N}_0$ and $q \in \mathbf{N}$. Then $\Phi_n(\mathcal{F}_n^{1,q}) = \mathcal{X}_n^{1,q}$.*

PROOF. From Theorems 2.4 and 4.2, we know that $\Phi_n(\mathcal{F}_n^{1,q}) \subset \mathcal{X}_n^{1,q}$. So it remains to verify the reverse inclusion.

Suppose that $\phi \in \mathcal{X}_n^{1,q}$. Then

$$\phi(\zeta, v) = \sum_{m=0}^{\infty} \phi_{(m)}(\zeta, v),$$

where each $\phi_{(m)}$ is a $\bar{\mathcal{P}}(n, \mathbf{C}^q)$-valued holomorphic polynomial that is homogeneous of degree m on $\mathbf{B}^q$. Since $0 = D\phi(\zeta, v) = \sum_{m=0}^{\infty} D\phi_{(m)}(\zeta, v)$, and since the operator D preserves the homogeneity, hence the linear independence of the functions $D\phi_{(m)}$, we conclude that $D\phi_{(m)} = 0$ for all $m \in \mathbf{N}_0$. The proof of Theorem 4.2 implies that any polynomial in ζ has finite norm. Hence $\phi_{(m)} \in \mathcal{X}_n^{1,q}$ for each m.

In the case that $d(n, q) > 1$, the results of [**4**] imply that each $\phi_{(m)}$ is also an element of $\Phi_n(\mathcal{F}_n^{1,q})$. In fact, this also holds in the line bundle case, i.e., when $d(n, q) = 1$. Corollary 2.6 shows that when $d(n, q) = 1$, the image of a monomial is a monomial. Thus in this case we can invert Φ_n directly from Corollary 2.6.

So, with $\phi_{(m)}$ as above, there exists $f_{(m)} \in \mathcal{F}_n^{1,q}$ with $\Phi_n f_{(m)} = \phi_{(m)}$ for each $m \in \mathbf{N}_0$. By Corollary 2.6, $f_{(m)}$ must have the form

$$f_{(m)}(z) = \sum_{\substack{\nu \in \mathbf{N}_0^q \\ |\nu| = m+n}} a_{m,\nu} z_1^{|\nu|-n} \bar{z}_S^{\nu}.$$

Put $f = \sum f_{(m)}$. Since $\sum_{m=0}^{N} \phi_{(m)}$ converges to ϕ uniformly on compact subsets of $\mathbf{B}^q$ (see (3.3)), we may apply the dominated convergence theorem as in 4.2, along with Corollary 4.3, to get

$$\begin{aligned} \|\phi\|_{n,q}^2 &= \lim_{N\to\infty} \| \sum_{m=0}^{N} \phi_{(m)} \|_{n,q}^2 \\ &= \lim_{N\to\infty} \langle \sum_{m=0}^{N} f_{(m)}, \sum_{m=0}^{N} f_{(m)} \rangle \\ &= \lim_{N\to\infty} \sum_{m=0}^{N} \sum_{\substack{\nu \in \mathbf{N}_0^q \\ |\nu| = m+n}} |a_{m,\nu}|^2 (|\nu| - n)! \nu! \\ &= \langle f, f \rangle. \end{aligned}$$

Hence $\langle f, f \rangle < \infty$, and so $f \in \mathcal{F}_n^{1,q}$. From the continuity of Φ_n we then conclude that $\Phi_n f = \phi$. ■

REFERENCES

1. V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform Part 1*, Comm. Pure Appl. Math. **14** (1961), 187–214.
2. R. Blattner and J. Rawnsley, *Quantization of the action of $U(k,l)$ on $\mathbf{R}^{2(k+l)}$*, J. Funct. Anal. **50** (1983), 188–214.
3. M. Davidson, *The harmonic representation of $U(p,q)$ and its connection with the generalized unit disk*, Pacific J. Math. **129, No. 1** (1987), 33–55.
4. M. Davidson, T. Enright, and R. Stanke, *Differential operators and highest weight representations*, Mem. Amer. Math. Soc. **455** (1991), 1–117.
5. S. Helgason, *Groups and Geometric Analysis*, Academic Press, Inc., New York, 1984.
6. L.K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains*, American Mathematical Society, Providence, 1963.
7. H. Jakobsen and M. Vergne, *Wave and Dirac operators and representations of the conformal group*, J. Funct. Anal. **24** (1977), 52–106.
8. M. Kashiwara and M. Vergne, *On the Segal-Shale-Weil representations and harmonic polynomials*, Invent. Math. **44** (1978), 1–47.
9. R.A. Kunze, *Positive definite operator-valued kernels and unitary representations*, Functional Analysis (Proceedings of a Conference held at the University of California, Irvine) (Gelbaum, ed.), Thompson Book Company, Washington, 1966, pp. 235–247.
10. L.A. Mantini, *An integral transform in L^2-cohomology for the ladder representations of $U(p,q)$*, J. Funct. Anal. **60, No. 2** (1985), 211–241.
11. L.A. Mantini, *An L^2-cohomology construction of negative spin mass zero equations for $U(p,q)$*, J. Math. Anal. Appl. **136** (1988), 419–449.
12. R. Penrose, *Twistor functions and sheaf cohomology*, Advances in Twistor Theory, Research Notes in Mathematics 37, Pitman, London, 1979, pp. 25–36.
13. I.I. Piatetskii-Shapiro, *Automorphic Functions and the Geometry of Classical Domains*, Gordon and Breach, New York, 1969.

14. J. Rawnsley, W. Schmid, and J. Wolf, *Singular unitary representations and indefinite harmonic theory*, J. Funct. Anal. **51** (1983), 1-114.
15. W. Rudin, *Function Theory on the Unit Ball of* $\mathbf{C}^n$, Springer-Verlag, New York, 1980.
16. S. Sternberg and J. Wolf, *Hermitian Lie algebras and metaplectic representations*, Trans. Amer. Math. Soc. **238** (1978), 1–43.
17. R.O. Wells, Jr., *Complex manifolds and mathematical physics*, Trans. Amer. Math. Soc. **1** (1979), 296–336.

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FOURIER INVERSION IN SEVERAL VARIABLES – A NEW LOOK AT AN OLD PROBLEM

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Dedicated to Ray A. Kunze

ABSTRACT. We describe some results which give necessary and sufficient conditions for the convergence at a preassigned point of the spherical partial sums (resp. integrals) of the Fourier series (resp. Fourier integral) in the class of piecewise smooth functions on Euclidean space. These are also formulated for orthogonal series expansions in the eigenfunctions of the Laplace operator of a rotationally invariant Riemannian manifold. The conditions are vacuous in one and two dimensions, but yield non-trivial examples of divergent Fourier series/integrals in higher dimensions, with respect to a preassigned point.

1. Introduction

The problem of pointwise Fourier inversion in several variables is more than fifty years old ([B1], [B2]). For periodic piecewise smooth functions on the real line, the simplest results of classical Fourier analysis imply that the partial sums converge at every point to the normalized value of the function. The same result holds for periodic piecewise smooth functions on the plane. When we pass to three dimensions or higher, this is false. We show below that, for the indicator function of a three-dimensional ball, the spherical partial sums computed at the center of the ball oscillate between two positive limits. This also holds for: a) the Fourier integral expansion with respect to all of $\mathbb{R}^3$; b) the Fourier sine series with zero boundary values on any containing cube; c) the radial Fourier-Bessel series with respect to any containing ball, d) the spherical harmonic expansion (on the surface of $\mathbf{S}^4$) and e) the hyperbolic Fourier transform on the hyperbolic

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space $\mathbf{H}^3$. If we consider the difference of two such functions we obtain a simple example of non-localization in three dimensions: the indicator function of an annular region is zero in a neighborhood of the origin, but the Fourier expansion diverges at the origin.

For more general piecewise smooth functions we find a condition for convergence at a pre-assigned point. This condition is necessary and sufficient for the Fourier integral expansion with respect to all of $\mathbb{R}^n$, as well as for the corresponding expansions in spherical harmonics or the hyperbolic space. In the case of Fourier series we are limited to the discussion of radial functions, where we obtain a necessary and sufficient for convergence at the center. The Fourier-Bessel expansion is generalized to eigenfunction expansions on a class of rotationally invariant Riemannian manifolds.

Our viewpoint stems from the idea that Fourier analysis is a particular expression of the spectral theory of the Laplace operator. In particular we always use spherical summation as a definition of convergence. It is an open question to determine if the above results are valid for other definitions of convergence.

2. Convergence of Fourier transforms

A function $f \in L^1(\mathbb{R}^n)$ is *piecewise smooth* with respect to $x \in \mathbb{R}^n$ if there exists a finite set $0 = r_0(x) < r_1(x) < \ldots < r_K(x)$ so that the mean value

$$r \to \bar{f}_x(r) \equiv \frac{1}{\omega_{n-1}} \int_{S^{n-1}} f(x + r\omega) d\omega$$

is smooth on each subinterval (r_{i-1}, r_i) with right and left limits $\bar{f}_x(r_i \pm 0)$ and is identically zero for $r \geq r_K(x)$.

The *smoothness index* of f at x is the integer defined by

$$j(f; x) = \max\{j : r \to \bar{f}_x(r) \text{ is differentiable of class } C^j\}.$$

In general we have $j(f; x) \geq -1$ for any piecewise smooth function. $j(f; x) = 0$ signifies continuity with a discontinuous derivative. The condition $j(f; x) = +1$ signifies a continuous first derivative only, etc. etc.

The Fourier transform and spherical partial sum of an integrable function are defined by

$$\hat{f}(\mu) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\langle \mu, x \rangle} f(x)\, dx$$

$$f_M(x) = \int_{|\mu| \leq M} e^{i\langle \mu, x \rangle} \hat{f}(\mu)\, d\mu.$$

If f has n derivatives in L^1 then it is a classical fact that $\lim_{M \uparrow \infty} f_M(x) = f(x)$. For completeness we give a quick proof in the appendix. If f has less smoothness, then we have the following result, which describes the convergence of the Fourier transform in terms of the smoothness index.

THEOREM 1. *Suppose that f is piecewise smooth with respect to $x \in \mathbb{R}^n$. If $n = 1, 2$ then $\lim_{M \uparrow \infty} f_M(x) = \bar{f}_x(0 + 0)$.*

If $n \geq 3$ *then* $\lim_{M\uparrow\infty} f_M(x)$ *exists iff* $r \to \bar{f}_x(r)$ *has* $[\frac{n-3}{2}]$ *continuous derivatives, i.e.* $j(f;x) \geq [(n-3)/2]$*; in this case* $\lim_{M\uparrow\infty} f_M(x) = \bar{f}_x(0+0)$*. Otherwise we have*

$$-\infty < \liminf_{M\uparrow\infty} M^{-\nu}[f_M(x) - \bar{f}_x(0+0)] < \limsup_{M\uparrow\infty} M^{-\nu}[f_M(x) - \bar{f}_x(0+0)] < \infty$$

where $\nu = \frac{n-5}{2} - j(f;x)$ *and* $j(f;x)$ *is the smoothness index defined above.*

Many authors have proved equiconvergence theorems which reduce a multi-dimensional Fourier expansion to a corresponding problem of one-dimensional Fourier analysis; see [CCTV] for some recent results in this direction. The above theorem is of a different character, dealing specifically with the dimensionality in the statement of the conditions for convergence. The next section clarifies the situation in terms of a canonical example.

2a. An example in three dimensions.

Before turning to the proof of theorem 1, we give a basic example in which all of the computations can be made explicit. This is the indicator function of a ball in $\mathbb{R}^3$:

$$f^a(x) = \begin{cases} 1 & \text{if } 0 \leq |x| \leq a \\ 0 & \text{otherwise}, \end{cases}$$

where a is any positive real number. The Fourier transform is

$$\hat{f}^a(\mu) = \frac{1}{(2\pi)^3} \int_{|x|\leq a} e^{-i\langle \mu, x\rangle}\, dx.$$

For $\mu = 0$ we have $\hat{f}^a(\mu) = \frac{1}{(2\pi)^3} \times \frac{4}{3}\pi a^3$, whereas for $\mu \neq 0$ we have

$$\begin{aligned} \hat{f}^a(\mu) &= \frac{1}{2\pi^2|\mu|} \int_0^a \sin(r|\mu|)\, r\, dr \\ &= -\frac{1}{2\pi^2}\Big(\frac{a\cos a|\mu|}{|\mu|^2} - \frac{\sin a|\mu|}{|\mu|^3}\Big) \\ &:= F^a(|\mu|). \end{aligned}$$

At the center of the ball, the spherical partial sum is

$$\begin{aligned} f_M^a(0) &= \int_{|\mu|\leq M} \hat{f}^a(\mu) d\mu \\ &= \int_0^M F^a(y) 4\pi y^2\, dy \\ &= \frac{2}{\pi} \int_0^M \Big(\frac{\sin ay}{y} - a\cos ay\Big)\, dy. \end{aligned}$$

The first term is a convergent improper integral and the second integral is explicitly evaluated. Thus we have

$$f_M^a(0) = 1 - \frac{2}{\pi} \sin Ma + o(1), \qquad M \to \infty.$$

In particular, we have the exact fluctuation of the partial sums at the center:

$$\liminf_{M\to\infty} f_M^a(0) = 1 - \frac{2}{\pi}, \qquad \limsup_{M\to\infty} f_M^a(0) = 1 + \frac{2}{\pi}$$

By analogy with elementary Fourier series, we may define the Gibbs constant as the difference between the upper and lower limits, normalized by the discontinuity of the function at the edge:

$$G = \limsup_{M\to\infty} f_M^a(0) - \liminf_{M\to\infty} f_M^a(0) = \frac{4}{\pi} = 1.2732...$$

In contrast with the ordinary Gibbs phenomenon of one-dimensional Fourier series, here we have the additional fact that the partial sums *diverge* at a point. This is impossible for the one-dimensional Fourier series of a piecewise smooth function.

The above example can be modified to demonstrate the failure of localization in three-dimensional Fourier analysis. Specifically, let $0 < a < b$ and let the indicator function of the annulus be defined by $f^{a,b} = f^b - f^a$. Following the computations above, we have

$$f_M^{a,b}(0) = \frac{2}{\pi}(\sin Ma - \sin Mb) + o(1), \qquad M \to \infty$$

so that the spherical partial sums at $x = 0$ have no limit when $M \to \infty$. But the function $f^{a,b}$ is zero in a neighborhood of the origin.

If one carries out the above computation in higher dimensions, one finds (in accordance with theorem 1), that the spherical partial sums satisfy

$$-\infty < \liminf_{M\to\infty} M^{\frac{3-n}{2}} f_M^a(0) < \limsup_{M\to\infty} M^{\frac{3-n}{2}} f_M^a(0) < \infty$$

2b. Proof of Theorem 1.

The proof depends on the integral representation of the spherical partial sum

$$f_M(x) = \omega_{n-1} \int_0^a \bar{f}_x(r) D_n^M(r)\, r^{n-1}\, dr$$

where the Dirichlet kernel $D_n^M(r)$ is defined as the integral of the complex exponentials over a ball:

$$D_n^M(|z|) = \frac{1}{(2\pi)^n} \int_{|\mu|\le M} e^{i\langle \mu, z\rangle}\, dz,$$

where we have specifically noted the radial dependence. This may be expressed in terms of a Bessel function by the formula

$$D_n^M(r) = C_n M^n \frac{J_{n/2}(Mr)}{(Mr)^{n/2}}$$

where the constant depends only on the dimension n. Using known properties of the Bessel functions, we have the following fundamental properties of the Dirichlet kernel [SW]:

$$D_n^M(0) = C_n M^n, \qquad M \to \infty$$

$$D_n^M(r) = C_n \frac{M^{\frac{n-1}{2}}}{r^{\frac{n+1}{2}}} \left[\sin(Mr - \theta_n) + O(\frac{1}{M})\right], \qquad r > 0, M \to \infty$$

$$D_n^M(r) = -\frac{1}{2\pi r}\frac{d}{dr} D_{n-2}^M(r), \quad n \geq 3, r > 0.$$

Using this, we can write

$$f_M(x) = \omega_{n-1} \int_0^a \bar{f}_x(r) D_n^M(r) r^{n-1}\, dr$$
$$= -\frac{\omega_{n-1}}{2\pi} \int_0^a r^{n-2} \bar{f}_x(r) \frac{d}{dr} D_{n-2}^M(r)\, dr.$$

We break up the integration on each sub-interval (r_{i-1}, r_i) and integrate by parts; thus

$$\int_{r_{i-1}}^{r_i} r^{n-2} \bar{f}_x(r) \frac{d}{dr} D_{n-2}^M(r)\, dr$$
$$= r^{n-2} \bar{f}_x(r) D_{n-2}^M(r)|_{r_{i-1}}^{r_i} - \int_{r_{i-1}}^{r_i} \frac{d}{dr}[r^{n-2} \bar{f}_x(r)] D_{n-2}^M(r)\, dr.$$

When we sum over i, the boundary terms simplify to the *jumps* of the mean value function, while the new integral corresponds to a lower-dimensional Fourier partial sum: defining $\delta \bar{f}_x(r) = \bar{f}_x(r+0) - \bar{f}_x(r-0)$, we have in detail

$$2\pi \int_0^a \bar{f}_x(r) r^{n-1} D_n^M(r) dr$$
$$= \sum_{i=1}^{K} \delta \bar{f}_x(r_i) r_i^{n-2} D_{n-2}^M(r_i) + \int_0^a \frac{d}{dr}[r^{n-2} \bar{f}_x(r)] D_{n-2}^M(r)\, dr.$$

If $\delta \bar{f}_x(r_i) \neq 0$ for some i, then the first term is non-zero and, from the asymptotic behavior of the Dirichlet kernel, dominates the asymptotic behavior of the second integral when $M \to \infty$. If, on the other hand, all of these jump terms are zero, then we may reduce to the consideration of the lower dimensional integrand. In particular if $n = 2m + 1$ is odd and $j(f, x) \geq m - 1 = [\frac{n-3}{2}]$, then $\delta \bar{f}_x(r_i) = 0$ together with all derivative terms of orders less than or equal to $m - 1$. By repeated integration-by-parts m times we arrive at a one-dimensional Fourier integral. By one-dimensional Fourier analysis this has a limit, which can be computed by inserting a smooth function. From the classical theory (see the proof in the appendix), this limit exists and equals $\bar{f}_x(0+0)$. If, on the other hand, $n = 2m + 2$ is even and $j(f; x) \geq m - 1 = [\frac{n-3}{2}]$ then $\delta \bar{f}_x(r_i) = 0$ together with all derivative terms of orders less than or equal to $m - 1$. By repeated integration-by-parts m times we arrive at a two-dimensional Fourier-Bessel integral for a radial function, for which the convergence is proved directly.

3. Convergence of Fourier series of periodic functions

Passing to the analogous case of Fourier series, we first examine some concrete examples in $\mathbb{R}^3$.

3a. Examples in three variables.

In this sub-section we consider the Fourier series of the radial function $f : \mathbb{R}^3 \to \mathbb{R}^1$ defined by

$$f^a(x) = \begin{cases} 1 & \text{if } 0 \le |x| \le a \\ 0 & \text{otherwise.} \end{cases}$$

Here $0 < a \le \pi$, so that we are looking at the indicator function of a ball which is inside (possibly touching the edges of) the basic cube.

The Fourier coefficients are computed easily by evaluating the integrals in spherical coordinates r, ϕ, θ as follows:

$$\begin{aligned} A_{\mathbf{n}} &= (\frac{1}{2\pi})^3 \int_{|\mathbf{x}| \le a} e^{-i\langle \mathbf{n}, \mathbf{x} \rangle} dx_1 dx_2 dx_3 \\ &= (\frac{1}{2\pi})^3 \int_0^a \int_0^\pi \int_{-\pi}^\pi e^{-i|n| r \cos\phi} r^2 \sin\phi \, d\theta \, d\phi \, dr. \end{aligned}$$

For $|\mathbf{n}| = 0$ the integral is easily seen to be $A_{\mathbf{0}} = \frac{a^3}{6\pi^2}$. For $|\mathbf{n}| \neq 0$ we may evaluate the integral by elementary calculus to obtain

$$\begin{aligned} A_{\mathbf{n}} &= \frac{1}{2\pi^2 |\mathbf{n}|} \int_0^a \sin(|\mathbf{n}| r) \, r \, dr \\ &= \frac{1}{2\pi^2 |\mathbf{n}|} (-\frac{a \cos |\mathbf{n}| a}{|\mathbf{n}|} + \frac{\sin |\mathbf{n}| a}{|\mathbf{n}|^2}). \end{aligned}$$

The partial sums of this series for $a = \pi$, $|\mathbf{n}| \le \sqrt{134}$ and $(x_1, x_2, x_3) = (x, 0, 0)$ have been graphed using Mathematica and are depicted in Figure 1. The partial sums for $|\mathbf{n}| \le \sqrt{155}$ are depicted in Figure 2.

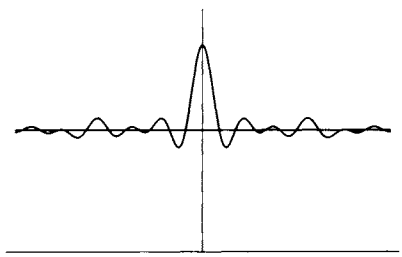

Figure 1

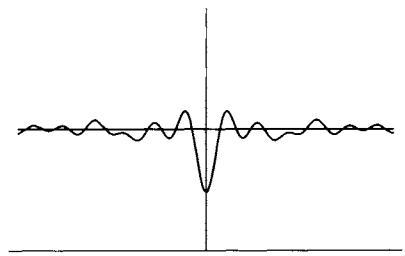

Figure 2

We note a dramatic oscillation in the neighborhood of $x = 0$, reminiscent of the Gibbs phenomenon from one-dimensional Fourier series. But in this case we have *divergence of the partial sums by oscillation*, which is totally unknown for one-dimensional Fourier series. The oscillation displayed by Figures 1 and 2 illustrates that some of the partial sums go above the line $y = 1$ while others go below. We prove in section 3b that the Fourier series diverges at $\mathbf{x} = \mathbf{0}$. However we have not been able to compute the Gibbs constant $\limsup_M f_M(0) - \liminf_M f_M(0)$.

This example may be modified to produce a simple example of the failure of localization in higher dimensions. We consider the indicator function of an annulus:

$$f(\mathbf{x}) = 1 \text{ for } a \leq |\mathbf{x}| \leq b \text{ and} \qquad f(\mathbf{x}) = 0 \text{ otherwise.}$$

Here $0 < a < b \leq \pi$. The function is zero in a neighborhood of $\mathbf{x} = 0$ but the Fourier series diverges by oscillation at $\mathbf{x} = 0$. Figures 3 and 4 are graphs of the partial sums of this series with $(x_1, x_2, x_3) = (x, 0, 0)$. In Figure 3, $a = \pi/4$, $b = \pi$ and $|\mathbf{n}| \leq \sqrt{134}$; in Figure 4, $a = 1$, $b = 3$ and $|\mathbf{n}| \leq \sqrt{125}$. We see a phenomenon similar to the one displayed in Figures 1 and 2; some of the partial sums are positive at $x = 0$ and others are negative.

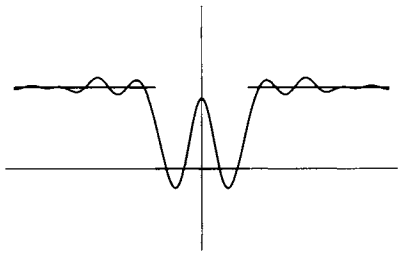

Figure 3

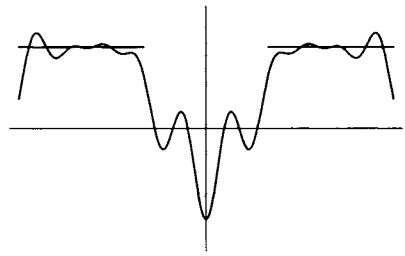

Figure 4

Previous examples of the failure of localization in low dimensions have depended on limiting procedures, such as infinite series, the principle of uniform boundedness or other non-constructive methods of functional analysis. We refer to the article of Ash [A] and the references cited therein.

3b. Proof of convergence/divergence.

The n-dimensional torus is $\mathbf{T}^n = \{x = (x_1, \ldots, x_n) : -\pi \leq x_i \leq \pi\}$. The Fourier coefficients and spherical partial sums of an integrable function $f(x)$, $x \in \mathbf{T}^n$ are

$$A_{\mathbf{m}} = \frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} f(x) e^{-i\langle \mathbf{m}, x\rangle}\, dx, \qquad \mathbf{m} \in \mathbb{Z}^n$$

$$f_M(x) = \sum_{|m| \leq M} A_m e^{i\langle m, x\rangle}, \qquad x \in \mathbb{R}^n$$

The spherical mean value is

$$\bar{f}_x(r) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} f(x + r\omega) d\omega,$$

where we extend the function f periodically to $\mathbb{R}^n$.

DEFINITION. $f \in L^1(\mathbf{T}^n)$ is a *piecewise smooth radial function with respect to* $x \in \mathbf{T}^n$ if $r \to \bar{f}_x(r)$ is piecewise C^∞ and the periodic extension satisfies $f(x + r\omega) = \bar{f}_x(r)$, $0 \leq r \leq \pi$, $\omega \in \mathbf{S}^{n-1}$ and $f(x + r\omega) = 0$ if $r > \pi$, $r\omega \in \mathbf{T}^n$.

In two dimensions the proof of convergence for a piecewise smooth radial function can be accomplished by appealing to a theorem of Hardy and Landau [HL]. Otherwise we proceed as follows; for a radial function the Fourier coefficient is the restriction of a smooth radial function: $A_m = A(|m|)$ where the asymptotic behavior of $A(r), A'(r), r \uparrow \infty$ is determined explicitly. The spherical partial sum is expressed as a Stieltjes integral:

$$f_M(0) := \sum_{|m| \leq M} A_m = \int_0^M A(r)\, dN(r)$$

where $N(r)$ is the counting function of lattice points, satisfying *Landau's estimate*

$$N(r) := \sum_{|m| \leq r} 1 = c_n r^n + O(r^{n-2+\frac{2}{n+1}}) \qquad r \to \infty.$$

This is used to write

$$f_M(0) - \int_0^M A(r) c_n\, n r^{n-1}\, dr$$

$$= A(M)[N(M) - c_m M^n] + \int_0^M A'(r)[N(r) - c_n r^n]\, dr$$

and leads to the following theorem, which extends the results in [PST].

THEOREM 2. *Suppose that f is piecewise smooth and radial with respect to $x \in \mathbf{T}^n$. If $n = 1, 2$ the Fourier series converges, and*

$$\lim_{M \to \infty} f_M(x) = \bar{f}_x(0 + 0).$$

If $n \geq 3$ the Fourier series converges iff the function $r \to \bar{f}_x(r)$ has at least $[\frac{n-3}{2}]$ continuous derivatives.

3c. Fourier series with boundary conditions.

The methods of the preceding section can be used to study the expansion of a radial function in an eigenfunction series in the cube $-\pi \leq x_i \leq \pi$:

$$f(x) \sim \sum_{m \in Z^n} A_m \phi_m(x)$$

where the eigenfunctions ϕ_m satisfy $\Delta\phi + \lambda\phi = 0$ in the cube and the Dirichlet condition that $\phi = 0$ on the boundary. (The corresponding problem with Neumann or mixed Dirichlet-Neumann boundary conditions can be dealt with by suitably modifications)

Separation-of-variables produces the eigenfunctions in the form

$$\phi_{m_1,\dots,m_n}(x_1,\dots,x_n) = \Pi_{i=1}^n \left(\cos\frac{m_i x_i}{2}\sin\frac{m_i\pi}{2} + \sin\frac{m_i x_i}{2}\cos\frac{m_i\pi}{2}\right).$$

The Fourier coefficients of a radial function $f(x) = F(|x|)$ are computed as

$$A_{m_1,\dots,m_n} = \pi^{-n}\int_{|x|\le\pi} F(|x|)\,\phi_{m_1,\dots,m_n}(x_1,\dots,x_n)\,dx.$$

At the center of the cube we have

$$\sum_m A_m\phi_m(0) = \sum_{m_i \text{ odd}} A_m \Pi_{i=1}^n \sin\frac{m_i\pi}{2}.$$

But for m_i odd,

$$A_m = \pi^{-n}\int_{|x|\le\pi} F(|x|)\,\Pi_{i=1}^n \cos\frac{m_i x_i}{2}\sin\frac{m_i\pi}{2}\,dx$$

so that

$$\sum_m A_m\phi_m(0) = \pi^{-n}\sum_m \int_{|x|\le\pi} F(|x|)\Pi_{i=1}^n \cos\frac{m_i x_i}{2}\sin^2\frac{m_i\pi}{2}\,dx.$$

To complete the analysis of the Fourier expansion, we need the asymptotic behavior of the number of *positive* lattice points, defined as

$$N_n^+(\lambda) := \sum_{m_i>0,|m|\le\lambda} 1.$$

PROPOSITION. *When* $\lambda\to\infty$, $N_n^+(\lambda) = e_n\lambda^n - f_n\lambda^{n-1} + O(\lambda^{n-2+\frac{2}{n+1}})$, *for suitable positive constants* e_n, f_n.

A corresponding analysis leads to the following identical result:

THEOREM 3. *Suppose that* $f(x) = F(|x|)$ *where* F *is piecewise smooth on* $[0,\pi]$ *and zero elsewhere. Then the series expansion* $\sum_m A_m\phi_m(0)$ *converges iff* F *is of class* $[\frac{n-3}{2}]$.

4. Spherical harmonic expansions

The expansion of an L^2 function on the sphere $\mathbf{S}^n$ is written

$$f \sim \sum_{m=0}^{\infty} Y_m^{(n)}$$

where $Y_m^{(n)} \in \mathcal{H}_m$, the space of spherical harmonics of degree m with respect to the sphere $\mathbf{S}^n$. The partial sum $f_M := \sum_{0 \le m \le M} Y_m^{(n)}$ is the orthogonal projection of f onto the linear subspace of spherical harmonics of degree $\le M$. The spherical harmonics of degree m are $SO(n+1)$-invariantly defined as homogeneous polynomials of degree m which satisfy Laplace's equation $\Delta_{R^{n+1}} Y = 0$. In order to compute $f_M(Q)$, we may take a basis of spherical harmonics with respect to the axis $Q \in \mathbf{S}^n$. Without loss of generality we will suppose that $Q = (1, 0, \dots, 0)$ is a fixed origin from which distance on the sphere will be measured. A system of geodesic polar coordinates is written (r, θ) , where $0 \le r \le \pi$ and $\theta \in \mathbf{S}^{n-1}$. With this notation the metric of $\mathbf{S}^n$ is written

$$ds^2 = dr^2 + \sin^2 r \, d\theta^2$$

and the Laplacian is written

$$\Delta_{\mathbf{S}^n} = \frac{d^2}{dr^2} + (n-1) \cot r \frac{d}{dr} + \frac{1}{\sin^2 r} \Delta_{\mathbf{S}^{n-1}}$$

An n-dimensional spherical harmonic is written

$$Y_m^{(n)}(r, \theta) = \phi(r) \psi(\theta)$$

where the radial factor ϕ is a solution of the ordinary differential equation

$$\phi'' + (n-1) \cot r \, \phi' + (\lambda - \frac{\beta}{\sin^2 r}) \phi = 0 \qquad \lambda = |\mathbf{m}|(|\mathbf{m}| + n - 1)$$

and $\psi(\theta)$ is a spherical harmonic on $\mathbf{S}^{n-1}$, solution of

$$\Delta_{\mathbf{S}^{n-1}} \psi + \beta \psi = 0.$$

An integrable function $f(Q)$, $Q \in \mathbf{S}^n$ is said to be *piecewise smooth* if for all $Q_0 \in \mathbf{S}^n$ the spherical mean value

$$(0, \pi) \ni r \to \bar{f}_{Q_0}(r) = \frac{1}{\omega_{n-1}} \int_{\{Q : d(Q, Q_0) = r\}} f(Q) \, dS(Q)$$

is piecewise C^∞ on the interval $0 \le r \le \pi$. The Fourier expansion of a piecewise smooth function is written

$$f \sim \sum_{k,m} A_{k,m} \phi_{k,m}(r) \psi_m(\theta)$$

where

$$A_{k,m} = \frac{\int_{\mathbf{S}^{n-1}} \int_0^\pi f(r, \theta) \phi_{k,m}(r) \psi_m(\theta) \sin^{n-1} r \, dr \, d\omega}{\int_{\mathbf{S}^{n-1}} \int_0^\pi \phi_{k,m}(r)^2 \psi_m(\theta)^2 \sin^{n-1} r \, dr \, d\omega}.$$

From the above remarks, we have the series at $r = 0$ in the form

$$f(0) \sim \sum_{k \ge 1} A_{k,0} \phi_{k,0}(0)$$

and the f-dependence of the Fourier coefficient is given by

$$\int_{\mathbf{S}^{n-1}} \int_0^\pi f(r, \theta) \phi_{k,0}(r) \sin^{n-1} r \, dr \, d\omega = \omega_{n-1} \int_0^\pi \bar{f}_0(r) \phi_{k,0}(r) \sin^{n-1} r \, dr.$$

This shows that *the series for $f(0)$ depends only on the spherical mean value* $\bar{f}_0(r), 0 \le r \le \pi$. Therefore we may restrict attention to radial functions and obtain the following result.

THEOREM 4. *Let $f(Q), Q \in \mathbf{S}^n$ be a piecewise smooth function. Then the spherical harmonic expansion converges at $Q_0 \in \mathbf{S}^n$ iff the spherical mean $r \to \bar{f}_{Q_0}(r)$ has $[\frac{n-3}{2}]$ continuous derivatives.*

5. Dirichlet eigenfunction expansions

In the previous sections we presented results on the convergence of series expansions of functions on complete manifolds without boundary, where we have the full group of translations, allowing any point as the origin. When we pass to manifolds with boundary, we have a privileged origin, where we can study the convergence/divergence of the expansions. To motivate the exposition, we first present the case of a Euclidean ball with Dirichlet boundary conditions, where the analysis is explicitly described by Bessel functions. Previous studies of this case with graphical illustrations are contained in [GP]

5a. Analysis on $\mathbb{R}^n$: Fourier-Bessel series.

We consider the Euclidean space $\mathbb{R}^n$ with a ball of radius $a > 0$:

$$B_a = \{x : 0 \leq |x| < a\}$$

The n-dimensional Laplace operator is $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$.

Let $\phi = \phi_{k,\mathbf{m}}(x)$ denote the eigenfunctions of the n-dimensional Laplace operator in the ball:

$$\begin{aligned} \Delta\phi + \lambda\phi = 0, &\qquad 0 \leq |\mathbf{x}| < a \\ \phi(x) = 0, &\qquad |\mathbf{x}| = a. \end{aligned}$$

These may be computed in terms of Bessel functions and spherical harmonics by the technique of separation of variables according to

$$\phi_{k,\mathbf{m}}(\mathbf{x}) = r^{|\mathbf{m}|} \frac{J_{|\mathbf{m}|+\frac{1}{2}(n-2)}(r\sqrt{\lambda})}{r^{|\mathbf{m}|+\frac{1}{2}(n-2)}} Y_{\mathbf{m}}(\omega), \quad r = |\mathbf{x}|, \omega = \mathbf{x}/|\mathbf{x}|, \quad \lambda = [\frac{x_k^{(m,n)}}{a}]^2.$$

Here the first factor denotes the standard Bessel function and the second factor is a *spherical harmonic of degree* $|\mathbf{m}|$, a solution of the equation $\Delta_{S^{n-1}} Y + |\mathbf{m}|(|\mathbf{m}| + n - 2)Y = 0$, where $\Delta_{S^{n-1}}$ is the spherical Laplacian operator. The quantity $x_k^{(m,n)}$ denotes the k^{th} zero of the Bessel function $J_{|\mathbf{m}|+\frac{1}{2}(n-2)}$. The index $k = 1, 2, \ldots$ while the index $\mathbf{m}$ takes values in a finite set which labels the spherical harmonics of degree $|\mathbf{m}| = 0, 1, 2 \ldots$. These functions are orthogonal with respect to the Euclidean volume element in the ball B_a, which, when written in polar coordinates is

$$\begin{aligned} \langle \phi_{k_1,\mathbf{m}_1}, \phi_{k_2,\mathbf{m}_2} \rangle &= \int_{B_a} \phi_{k_1,\mathbf{m}_1}(\mathbf{x}) \phi_{k_2,\mathbf{m}_2}(\mathbf{x}) d\mathbf{x} \\ &= \int_0^a \int_{\mathbf{S}^{n-1}} \phi_{k_1,\mathbf{m}_1}(\mathbf{x}) \phi_{k_2,\mathbf{m}_2}(\mathbf{x}) r^{n-1}\, dr\, d\omega = 0, \quad (k_1, \mathbf{m}_1) \neq (k_2, \mathbf{m}_2). \end{aligned}$$

The Fourier-Bessel coefficients of an integrable function are defined by

$$A_{k,\mathbf{m}} = \frac{\int_{B_a} f(\mathbf{x}) \phi_{k,\mathbf{m}}(\mathbf{x}) d\mathbf{x}}{\int_{B_a} \phi_{k,\mathbf{m}}(\mathbf{x})^2 d\mathbf{x}}$$

THEOREM 5a. *Let $f(\mathbf{x})$, $\mathbf{x} \in B_a$, be an integrable function for which the mean value*

$$r \to \bar{f}(r) := \frac{\int_{\mathbf{S}^{n-1}} f(r\omega)d\omega}{\int_{\mathbf{S}^{n-1}} d\omega}$$

is piecewise C^∞.
i) If $n = 1, 2$, then $\lim_{M\to\infty} \sum_{\lambda_k \le M} A_{k,\mathbf{m}}\phi_{k,m}(0)$ exists.
ii) If $n \ge 3$ then $\lim_{M\to\infty} \sum_{\lambda_k \le M} A_{k,\mathbf{m}}\phi_{k,m}(0)$ exists if and only if the mean value function $\bar{f}(r)$ has $[\frac{(n-3)}{2}]$ continuous derivatives for $0 \le r \le a$ which vanish at $r = a$. ($[\ ]$ = integer part.)

REMARKS:. In case $n = 1$ the Fourier-Bessel series reduces to the ordinary Fourier series, which has been well-studied. In case $n = 2$ several authors have noted that, although the series converges at $r = 0$, the convergence may be expected to be slower than for $r > 0$. In fact, an explicit analysis of the Bessel function asymptotics reveals that for a piecewise smooth function with $\bar{f}(a) \neq 0$ the speed of convergence at $r = 0$ is of the order $O(M^{-1/2})$ whereas at $r > 0$ the speed of convergence is $O(M^{-1})$, just as for the one-dimensional Fourier series ([GO], p. 34). When we pass to the three-dimensional Fourier Bessel expansion of such a function, we find [GP] that the series diverges boundedly at $r = 0$, just as for the previous three-dimensional examples.

5b. General case: asymptotic solutions of differential equations.
We now analyze the eigenfunction expansions on a general manifold with rotational symmetry. The Riemannian metric is written in geodesic polar coordinates as

$$ds^2 = dr^2 + G(r)^2 d\theta^2$$

where $G(r)$ is a smooth function with $G(r) > 0$ $(r > 0), G(0) = 0, G'(0) = 1, G''(0) = 0$.

The equation satisfied by the radial eigenfunctions is

$$0 = \Delta\phi + \lambda\phi = \phi''(r) + (n-1)\frac{G'}{G}\phi'(r) + \lambda\phi = 0.$$

The asymptotic analysis for $\lambda \to \infty$ has been carried out by Olver [O, Theorem 4.1, p. 444]. With some work, these can be expressed asymptotically in terms of Φ_λ, which satisfies the Euclidean equation $\Phi'' + \frac{n-1}{r}\Phi' + \lambda\Phi = 0$ with $\Phi(0) = 1$. With a suitable normalization for ϕ_λ, we have

$$\phi_\lambda(r) = (\frac{G}{r})^{\frac{1-n}{2}}\Phi_\lambda(r) + \frac{1}{\sqrt{\lambda^\nu}}\frac{G^{\frac{1-n}{2}}}{\sqrt{r}}\epsilon(r^2, \lambda) \qquad \lambda \to \infty.$$

where $\lim_{\lambda\to\infty} \epsilon = 0$. In particular $\phi_\lambda(0) = 1 + O(\frac{1}{\sqrt{\lambda}}), \lambda \to \infty$. For fixed $r > 0$ the first term is $O(\frac{1}{\sqrt{\lambda}^{(\nu+\frac{1}{2})}})$ while the second term is $O(\frac{1}{\sqrt{\lambda}^{(\nu+1)}})$ when $\lambda \to \infty$. This leads to the following theorem, exactly as for the Fourier-Bessel case.

THEOREM 5B. *Let $f(r)$, $0 \le r \le a$ be a piecewise smooth radial function on a geodesic ball of an n-dimensional Riemannian manifold with rotational symmetry. Then the eigenfunction expansion $\sum_k A_k\tilde{\phi}_{\lambda_k}(0)$ converges iff $r \to f(r)$ has $[\frac{n-3}{2}]$ continuous derivatives.*

Appendix: proof of the classical result

For completeness, we give a quick proof of the classical convergence theorem of Fourier transforms of smooth functions on $\mathbb{R}^n$. Without loss of generality, we prove convergence at $x = 0$, which can always be arranged by a suitable translation.

We first prove convergence in case $f(x) = e^{-|x|^2/2}$. In this case a direct computation gives $\hat{f}(\mu) = e^{-|\mu|^2/2}/(2\pi)^{n/2}$, so that $f_M(0) \to \int_{R^n} \hat{f}(\mu)\, d\mu = 1$.

Next we consider $f^{(i)}(x) = x_i e^{-|x|^2/2}$, $1 \leq i \leq n$. In this case a direct computation gives $\hat{f}^i(\mu) = \mu_i e^{-|\mu|^2/2}/(2\pi)^{n/2}$, so that $f^i_M(0) = 0$ for all M and the required convergence also holds.

Next, consider any integrable function for which $f(x) = O(|x|^2), |x| \to 0$. Define $h(x) = f(x)/|x|^2$, also an integrable function. Its Fourier transform satisfies
$\hat{h}(\mu) = (\frac{1}{2\pi})^n \int e^{-i\langle \mu, x\rangle}\, h(x)dx$, a C^2 function, whose Laplacian is

$$\Delta\hat{h}(\mu) = -\frac{1}{(2\pi)^n} \int |x|^2 e^{\langle -i\mu, x\rangle}\, h(x)dx = -\hat{f}(\mu).$$

The spherical partial sum is transformed by the divergence theorem as follows:

$$\begin{aligned} f_M(0) &= \int_{|\mu|\leq M} \hat{f}(\mu)d\mu \\ &= -\int_{|\mu|\leq M} \Delta\hat{h}(\mu)d\mu \\ &= -\int_{|\mu|=M} \frac{\partial\hat{h}}{\partial n}(\mu)\, dS \to 0 \end{aligned}$$

where we have used the smoothness of f in the last step.

Finally, for the general case, write

$$f(x) = f(0)e^{-|x|^2/2} + \sum_{i=1}^{n} \partial_i f(0) x_i e^{-|x|^2/2} + F(x)$$

which defines F. Clearly F is also smooth and satisfies $F(0) = 0$, $\partial_i F(0) = 0$; hence $F(x) = O(|x|^2), |x| \to 0$. From the above argument we conclude that $F_M(0) \to 0$ when $M \to \infty$, hence $f_M(0) \to f(0)$, which was to be proved.

References

[A] M. Ash, *Multiple trigonometric series*, In: *Studies in Harmonic Analysis*, Studies in Mathematics, MAA, Washington DC.

[B1] S. Bochner, *Ein Konvergensatz für mehrvariablige Fouriersche Integrale,*, Math. Z. **34** (1931), 440 - 447.

[B2] S. Bochner, *Summation of multiple Fourier series by spherical means*, Tran. Amer. Math. Soc. **40** (1936), 175 - 207.

[CCTV] L. Colzani, A. Crespi, G. Travaglini and M. Vignati, *Equiconvergence theorems for Fourier-Bessel expansions with applications to the harmonic analysis of radial functions in euclidean and non-euclidean spaces*, Trans. Amer. Math. Soc. **338** (1993), 43 - 55.

[GO] D. Gottlieb and S. Orzag, *Numerical Analysis of Spectral Methods: Theory and Applications*, SIAM, Philadelphia, PA, 1977.

[GP] A. Gray and M. Pinsky, *Gibbs' phenomenon for Fourier-Bessel Series*, Expositiones Mathematicae **11** (1993), 123 – 135.

[HL] G.H. Hardy and E. Landau, *The lattice points of a circle,*, Proc. Royal Soc., A, **105** (1924), 244 – 258.

[O] F. W. Olver, *Asymptotics and Special Functions*, Academic Press, New York, 1974.

[P] M. Pinsky, *Pointwise Fourier inversion and related eigenfunction expansions*, Commun. Pure Appl. Math. **47** (1994), 653 – 681.

[PST] M. Pinsky, N. Stanton and P. Trapa, *Fourier series of radial functions in several variables*, J. Functional Analysis **116** (1993), 111-132.

[SW] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis in Euclidean Spaces*, Princeton University Press, Princeton, N.J., 1971.

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Total Positivity, Harmonic Analysis and Random Walks on Weyl Chambers

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Dedicated to Ray A. Kunze

ABSTRACT. This paper continues our study of total positivity from a group-theoretic perspective. In earlier investigations we used techniques from harmonic analysis on compact Lie groups to establish total positivity of functions. In the present paper we indicate how the theory of random walks on Weyl chambers leads to combinatorial examples of totally positive functions and, more generally, how totally positive functions arise naturally in representation theory.

1. Introduction

This article is the third in a series of papers treating the concept of total positivity from the point of view of group theory and harmonic analysis.

In the first paper [6], we related the classical notion of total positivity to analysis on the unitary groups $\mathrm{U}(n)$ and applied our techniques to obtain new positivity theorems for classical hypergeometric functions. In the second paper [7] we considered finite reflection groups and associated with each such group a new notion of total positivity. In contrast to the previous papers in which the results were primarily analytic in nature, the present paper has a combinatorial flavor. Here, we use the idea of a random walk on a Weyl chamber – a concept

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that arises in statistics in the study of sequential data analysis – and a deep combinatorial formula due to Gessel and Zeilberger [4], to construct totally positive functions.

By way of background, we briefly review the classical notion of total positivity and its connection with the unitary groups.

Let $\mathcal{D} \subseteq \mathbb{R}^2$ and r a positive integer. A function $K : \mathcal{D} \to \mathbb{R}$ is said to be *totally positive of order* r (TP_r) if the $n \times n$ determinants

$$\det(K(s_j, t_k)) = \begin{vmatrix} K(s_1,t_1) & K(s_1,t_2) & \cdots & K(s_1,t_n) \\ K(s_2,t_1) & K(s_2,t_2) & \cdots & K(s_2,t_n) \\ \vdots & \vdots & & \vdots \\ K(s_n,t_1) & K(s_n,t_2) & \cdots & K(s_n,t_n) \end{vmatrix} \tag{1}$$

are nonnegative for all $n = 1, \ldots, r$ and for all $s_1 > \cdots > s_r$ and $t_1 > \cdots > t_r$ such that $(s_j, t_k) \in \mathcal{D}$. If the determinants (1) are all strictly positive, then one says that K is *strictly totally positive of order* r (STP_r). If K is TP_r (STP_r) for all $r = 1, 2, \ldots$, then we say that K is TP_∞ (STP_∞). We mention in passing that there is a still stronger notion of total positivity called *extended total positivity* (ETP) which can be found, for example, in [6], [12].

Total positivity arises historically in the 1930's in work of Schoenberg [16] and Gantmacher and Krein [2,3] on oscillation properties of integral operators, Sturm-Liouville theory, and the analysis of mechanical vibrations. It also has widespread applications in probability and statistics and in approximation theory. The principal reference is Karlin's comprehensive treatise [12].

The central theme of our work in [6] is the intimate relationship between total positivity of multiplicative real-analytic kernels – that is, functions of the form $K(x, y) = f(xy)$ where $f : \mathbb{R} \to \mathbb{R}$ is real-analytic – and analysis on the unitary groups. In particular, to any such kernel K we associate a sequence of real-valued functions ψ_n of a matrix argument, having the property that for each $n = 1, 2, \ldots$

$$\det(K(s_j, t_k)) = V(s)V(t) \int_{\mathrm{U}(n)} \psi_n(usu^{-1}t)\, du. \tag{2}$$

Here, du denotes Haar measure on the unitary group $\mathrm{U}(n)$ normalized to have total volume one; $s = \mathrm{diag}(s_1, \ldots, s_n)$ and $t = \mathrm{diag}(t_1, \ldots, t_n)$ are diagonal matrices such that all products $s_j t_k$ $(j, k = 1, \ldots, n)$ lie in the domain of f; and $V(s) = \prod_{1 \le j < k \le n}(s_j - s_k)$ denotes the Vandermonde determinant. It follows from (2) that the function K is totally positive of order r if the functions ψ_n are nonnegative for $n = 1, \ldots, r$. Conversely it can be shown that the total positivity of K implies that the functions ψ_n are nonnegative. Thus the integral representation (2) yields necessary and sufficient conditions for the total positivity of K [6].

The context in which f is a classical hypergeometric function is particularly fruitful, as the functions ψ_n are hypergeometric functions of matrix argument

[6,7] which, for appropriate values of the parameters and argument, can be shown to be positive. For example, when f is the exponential function, $f(x) = e^x$, the associated functions are given by $\psi_n(\xi) = \beta_n e^{\operatorname{tr}\xi}$ for all $n \times n$ complex matrices ξ, where $\beta_n = \prod_{j=1}^{n}(j-1)!$, a positive constant. Here, (2) takes the form

$$\int_{\mathrm{U}(n)} e^{\operatorname{tr}(usu^{-1}t)}du = \beta_n \frac{\det(e^{s_j t_k})}{V(s)V(t)} \tag{3}$$

from which we see immediately that the function $K(x,y) = e^{xy}$ is STP_∞. This is perhaps the most basic example of a totally positive function.

As we will see in the next section, the association of the unitary groups with the classical notion of total positivity is far from accidental. Indeed, from an expanded Lie-theoretic perspective, the notion of total positivity is intrinsically group theoretic and formula (3) is the special case for the group $\mathrm{U}(n)$ of a general formula ((10), below) due to Harish-Chandra that is valid for all connected compact Lie groups.

2. Total positivity for finite reflection groups

Let W be a finite reflection (or Coxeter) group acting on $\mathbb{R}^n$, by which one means a finite subgroup of the orthogonal group $\mathrm{O}(n)$ acting naturally on $\mathbb{R}^n$ by orthogonal linear transformations. We denote the action of $w \in W$ on $s \in \mathbb{R}^n$ by $w \cdot s$, and remark that $\det w = \pm 1$ for all $w \in W$. The action of W partitions $\mathbb{R}^n$ into a finite number of mutually congruent open subsets known as *Weyl chambers*. The boundary of a Weyl chamber is a union of hyperplanes called *walls* of the chamber. The group W then acts simply transitively on the set of Weyl chambers. We choose a fixed chamber $\mathcal{C}$ and refer to it as the *fundamental* Weyl chamber. For a complete treatment of finite reflection groups, and in particular a more explicit definition of the fundamental Weyl chamber in terms of a system of *simple roots*, see [9].

2.1. DEFINITIONS. Let $\mathcal{D}$ be a subset of $\mathbb{R}^n \times \mathbb{R}^n$ closed under the action of $W \times W$, i.e. $(s,t) \in \mathcal{D}$ implies $(w_1 s, w_2 t) \in \mathcal{D}$ for all $w_1, w_2 \in W$.

(i) A function $K : \mathcal{D} \to \mathbb{R}$ is *invariant with respect to* W, or simply W-*invariant*, if $K(w \cdot s, w \cdot t) = K(s,t)$ for all $w \in W$ and all $(s,t) \in \mathcal{D}$.

(ii) For any W-invariant function $K : \mathcal{D} \to \mathbb{R}$ set

$$D_W K(s,t) = \sum_{w \in W} (\det w) K(w \cdot s, t) \tag{4}$$

for all $(s,t) \in \mathcal{D}$,

(iii) A W-invariant function $K : \mathcal{D} \to \mathbb{R}$ is *totally positive with respect to* W if

$$D_W K(s,t) \geq 0 \tag{5}$$

for all $(s,t) \in \mathcal{D} \cap (\mathcal{C} \times \mathcal{C})$, where $\mathcal{C}$ is the fundamental chamber. If strict inequality holds in (5) we will say that K is *strictly totally positive with respect to* W.

The alternating sum D_W is an analog for W of the usual determinant. In fact, when

$$K(s,t) = \prod_{j=1}^{n} \tilde{K}(s_j, t_j) \tag{6}$$

for some function $\tilde{K}$ of two real variables, then (4) takes the form

$$D_W K(s,t) = \sum_{w \in W} (\det w) \prod_{j=1}^{n} \tilde{K}((w \cdot s)_j, t_j), \tag{7}$$

which is more closely analogous to the ordinary determinant. Indeed, when W is the symmetric group, (7) is precisely $\det(\tilde{K}(s_i, t_j))$. In [7] we restricted our attention to functions K of the form (6) for which the definition of $D_W K$ in (7) was sufficient. However, for the functions treated in this paper the definition (4)is more appropriate. In the context of total positivity this definition was first written down by Karlin and Rinott [13], but the concept of an alternating sum is of longstanding importance in representation theory (cf. Section 4 below).

The proposition that follows provides analogues for D_W of three basic properties of the ordinary determinant: (i) If the rows or columns of a determinant are permuted then the sign of the determinant changes in accordance with the sign of the permutation; (ii) The determinant is identically zero if two of its rows or columns are identical; (iii) The Binet-Cauchy formula. The first two parts appear as Proposition 3.2 in [8]. Part (iii), in various forms, appears in [8], [12], [13] and [17].

2.2. PROPOSITION. *(i) If $w \in W$ and $(s,t) \in \mathcal{D}$ then*

$$D_W K(s, w \cdot t) = D_W K(w \cdot s, t) = (\det w) D_W K(s,t).$$

(ii) If s or t belongs to a wall of a Weyl chamber then $D_W K(s,t) = 0$.

(iii) Let L and M be W-invariant functions on $\mathcal{D}_L = \mathcal{D}_1 \times \mathcal{D}_2$ and $\mathcal{D}_M = \mathcal{D}_2 \times \mathcal{D}_3$, respectively. Let μ be a W-invariant Borel measure on $\mathbb{R}^n$. Define

$$K(s,t) = \int_{\mathcal{D}_2} L(s,z) M(z,t) d\mu(z) \tag{8}$$

on the set $\mathcal{D}$ of all $(s,t) \in \mathbb{R}^n \times \mathbb{R}^n$ for which the integral is absolutely convergent. Then $\mathcal{D}$ is closed under the action of $W \times W$, K is W-invariant, and

$$D_W K(s,t) = \int_{\mathcal{D}_2 \cap C} D_W L(s,z) D_W M(z,t) d\mu(z) \tag{9}$$

for all $(s,t) \in \mathcal{D}$. In particular, if L and M are (strictly) totally positive with respect to W then so is K.

A large class of finite reflection groups, called *Weyl groups*, arises in the structure theory of compact Lie groups. For such finite reflection groups, harmonic analysis on the compact group provides a tool for constructing totally positive functions.

Thus, let U be a compact connected Lie group, $\mathfrak{u}$ its Lie algebra, $\mathfrak{g} = \mathfrak{u} + \sqrt{-1}\mathfrak{u}$ the complexification of $\mathfrak{u}$, and $B_0(\cdot,\cdot)$ the *trace form* on $\mathfrak{g}$. Let T denote the maximal torus of U, $\mathfrak{t}$ the Lie algebra of T, $\mathfrak{h} = \mathfrak{t} + \sqrt{-1}\mathfrak{t}$ the complexification of $\mathfrak{t}$, and $\mathfrak{h}^*$ the dual space of $\mathfrak{h}$. Let $\Delta \subset \mathfrak{h}^*$ be the root system for U, fix a basis of simple roots, and denote by Δ_+ the resulting subset of positive roots. The Weyl group W of U is the group generated by the set of simple reflections with respect to the roots. If we decompose $\mathfrak{t}$ as $\mathfrak{t} = \mathfrak{z} \bigoplus \mathfrak{t}_0$ where $\mathfrak{z}$ is the center of $\mathfrak{u}$ and $\mathfrak{t}_0$ is the semisimple part of $\mathfrak{t}$, then we can view W as a finite reflection group of orthogonal linear transformations on the real vector space $\mathfrak{t}_0$ (with respect to an inner product determined by the trace form). The dimension n of $\mathfrak{t}_0$ is called the *rank* of U. We fix as the fundamental Weyl chamber $\mathcal{C}$ in $\mathfrak{t}_0$ that chamber which is positive with respect to the system of simple roots.

We can now state the following formula of Harish-Chandra [10] that generalizes formula (3) above.

2.3. THEOREM. (Harish-Chandra) *Suppose that $H_1, H_2 \in \mathfrak{h}$ and du denotes Haar measure on U with total volume 1. Then*

$$\int_U e^{B_0(\mathrm{Ad}(u)\cdot H_1, H_2)}\, du = \beta_W \frac{\sum_{w\in W} (\det w) e^{B_0(w\cdot H_1, H_2)}}{V(H_1)V(H_2)} \tag{10}$$

where β_W is a positive constant, and

$$V(H) = \prod_{\alpha\in\Delta_+} \alpha(H)$$

for any $H \in \mathfrak{h}$.

2.4. COROLLARY. ([8], Theorem 6.1) *Suppose that the finite reflection group W is the Weyl group for a compact Lie group U. Then the function $K(s,t) = e^{\langle s,t\rangle}$, $s, t \in \mathbb{R}^n$, is strictly totally positive with respect to W.*

To prove this result, simply identify elements $H_1, H_2 \in \sqrt{-1}\mathfrak{t}_0$ with vectors $s, t \in \mathbb{R}^n$. If $s, t \in \mathcal{C}$ then, by Harish-Chandra's theorem,

$$\begin{aligned} D_W K(s,t) &= \sum_{w\in W} (\det w) K(w\cdot s, t) \\ &= \sum_{w\in W} (\det w) e^{\langle w\cdot s, t\rangle} \\ &= \sum_{w\in W} (\det w) e^{B_0(w\cdot H_1, H_2)} \\ &= \beta_W{}^{-1} V(H_1)V(H_2) \int_U e^{B_0(\mathrm{Ad}(u)\cdot H_1, H_2)} du. \end{aligned} \tag{11}$$

It is now evident that $D_W K(s,t) > 0$ for all $s, t \in \mathcal{C}$. Therefore, K is strictly totally positive with respect to W.

If we define total positivity with respect to formula (7) (rather than (4)) then we can rephrase the corollary as the statement that the function $\tilde{K}(x,y) = e^{xy}$

for $(x, y) \in \mathbb{R}^2$ is totally positive, not just in the classical sense, but also with respect to any Weyl group W.

2.5. EXAMPLES. (1) When $U = \mathrm{U}(n)$ the Weyl group W is S_n, the symmetric group on n symbols, and total positivity is the classical notion.

(2) For the orthogonal group $U = \mathrm{SO}(2n+1)$ the Weyl group W is generally denoted by $G(n)$ and can be described as follows: Each $w \in W$ is of the form $w = \epsilon\sigma$, where $\sigma \in S_n$ is a permutation on n symbols, $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ with $\epsilon_j = \pm 1$ for all $j = 1, \ldots, n$, and $w = \epsilon\sigma$ acts on $s = (s_1, \ldots, s_n) \in \mathfrak{h}$ by $w \cdot s = (\epsilon_1 s_{\sigma(1)}, \ldots, \epsilon_n s_{\sigma(n)})$. In particular, $\det(w) = \epsilon_1 \cdots \epsilon_n (\operatorname{sgn}\sigma)$.

For $U = \mathrm{SO}(2n+1)$ we identify $\mathfrak{t}_0$ with $\mathbb{R}^n$, and $\mathcal{C}$ with the collection of elements $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$ such that $s_1 > s_2 \cdots > s_n > 0$. Let $s = (s_1, \ldots, s_n), t = (t_1, \ldots, t_n) \in \mathbb{R}^n$. Upon simplifying the sum over W on the right-hand side of (11), we deduce ([8], Section 6) that the strict total positivity of the function $K(s,t) = e^{\langle s,t\rangle}$ is equivalent to positivity of the $n \times n$ determinant $\det(\sinh(s_j t_k))$ for all $s_1 > \cdots > s_n > 0$ and $t_1 > \cdots > t_n > 0$. Therefore the strict total positivity of the function K is equivalent to the well-known result [12] that the function $\tilde{K}(x,y) = \sinh(xy)$, $x, y > 0$, is strictly totally positive in the classical sense.

2.6 CONJECTURE. *We conjecture that the function $K(s,t) = e^{\langle s,t\rangle}$, $s, t \in \mathbb{R}^n$, is strictly totally positive with respect to all finite reflection groups.*

In view of Corollary 2.4, only the case in which W is not a Weyl group remains to be discussed. The following example indicates the difficulties to be addressed in resolving this conjecture.

Let $W = H_n^2$, the dihedral group with n reflections, acting on $\mathbb{R}^2$. Although W is a finite reflection group, only for $n = 1, 2, 3, 4$ and 6 is W the Weyl group of a compact Lie group. Working with polar coordinates on $\mathbb{R}^2$, we choose as the fundamental Weyl chamber the region

$$\mathcal{C} = \left\{ (r,\theta) : r > 0, \frac{(n-2)\pi}{2n} < \theta < \frac{\pi}{2} \right\}.$$

Then the criterion that the function K be strictly totally positive with respect to W reduces ([8], Section 6) to the inequality

$$\sum_{j=1}^{\infty} j I_{jn}(r) \sin j\theta > 0 \tag{12}$$

for $0 < \theta < \pi$, $r > 0$. Here, $I_j(r)$ denotes the Bessel function of imaginary argument and of order j.

The inequality (12) remains unproved for general n. However, the inequality is valid for $n = 1, 2, 3, 4, 6$ since, for those values of n, the dihedral group is the Weyl group of a compact Lie group, so that Corollary 2.4 applies. Moreover, the cases $n = 1, 2$ have also been established using classical results for the Bessel functions.

3. Random walks on Weyl chambers

The study of random walks and lattice paths is an old and venerable area of probability, statistics and combinatorics. In statistics, random walk problems arise in *sequential binomial sampling.* A lucid description of these problems, and their application to estimating the size of certain animal populations, is provided by Lehmann ([15], p. 92).

In the classical *ballot problem,* it is required to determine the probability that at no point in time did the winning candidate in a two-candidate electoral contest trail the loser [1]. This problem is solved by counting the number of ways of walking from the origin to a lattice point (m_1, m_2) with $m_1 > m_2$, taking positive unit-length steps in such a way that the walker always remains in the region $\{s_1 \geq s_2\}$. Here, each vote for the winner is identified with a horizontal step while each vote for the loser is identified with a vertical step.

More general ballot problems require enumeration of the number of ways of walking from the origin $(0, \ldots, 0)$ to a point $(m_1, \ldots, m_n)$, with steps of length 1 in the positive coordinate directions, and such that all steps lie in the region $s_1 \geq \cdots \geq s_n \geq 0$. This problem has been solved by several authors in various ways. A particularly elegant solution has been given by Zeilberger [18]. The result is as follows.

3.1. THEOREM. (Frobenius; MacMahon; Zeilberger) *The number of lattice paths from the origin to the point $m = (m_1, \ldots, m_n)$ taking positive unit-length steps, and with travel restricted to the region $s_1 \geq \cdots \geq s_n \geq 0$, is*

$$F(m) = (m_1 + \cdots + m_n)! \, \frac{\prod_{i<j}(m_i - m_j - i + j)}{\prod_{j=1}^{n}(m_j + n - j)!}. \tag{13}$$

The appearance of these numbers, which are familiar in representation theory as Weyl's degree formula, may seem surprising in this context. One explanation for their appearance begins with an analysis of the special case $n = 2$. In that case it may be verified using a simple graphical argument that $F(m_1, m_2)$ satisfies the recurrence relation

$$F(m_1, m_2) = \begin{cases} F(m_1 - 1, m_2) + F(m_1, m_2 - 1), & m_1 > m_2 \\ F(m_1, m_2 - 1), & m_1 = m_2 \end{cases}$$

Starting with the initial condition $F(0, 0) = 1$ the recurrence relation can be solved readily to yield the result in Theorem 3.1.

This technique extends to the case of general n. Indeed, if $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $e_n = (0, \ldots, 0, 1)$ denotes the standard basis for $\mathbb{R}^n$, then the number of paths $F(m)$ satisfies the recurrence relations

$$F(m) = \sum_j F(m - e_j) \tag{14}$$

where the sum is taken over all j such that $m - e_j$ remains in the region $s_1 \geq \cdots \geq s_n \geq 0$. As before, we also have the boundary condition $F(0, \ldots, 0) = 1$.

As it turns out, these recurrence relations are implicit in the work of Johnson [11], arising from the methods applied in [11] to decompose tensor products of the elementary representations of $\mathrm{GL}(n,\mathbb{C})$ (cf. Zhelobenko [19]). Specifically, it may be deduced from [11] that the recurrence relations (14) characterize the degrees of the irreducible polynomial representations of $\mathrm{GL}(n,\mathbb{C})$. Therefore the numbers $F(m)$ are given explicitly by (13).

It appears that these remarks constitute a new solution of the ballot problem, admittedly less elegant than the solution given by Zeilberger [18].

We turn now to the problem of constructing totally positive functions through the technique of ballot problems. Here, we shall apply a deep combinatorial formula of Gessel and Zeilberger [4] and will work within the context of abstract root systems.

Thus, following [4] and [5], let $\Delta \subset \mathbb{R}^n$ be a finite root system, W be its associated finite reflection group, Ψ be a base for Δ, and denote by $\mathcal{C}$ the fundamental Weyl chamber. Let L be a lattice embedded in $\mathbb{R}^n$, and assume that L is invariant under W; i.e., $w \cdot L = L$ for all $w \in W$. Denote by S a set of *allowable* steps, by which we mean a finite subset S of L that is also invariant under W.

We also assume that walks are *reflectable*, meaning that the following two conditions hold: (1) Any step $s \in S$ from a lattice point in the interior of $\mathcal{C}$ cannot exit $\mathcal{C}$; and (2) for each simple root $\alpha \in \Psi$, there is a real number k_α such that $\langle \alpha, s \rangle = \pm k_\alpha$ or 0 for all $s \in S$, and $\langle \alpha, l \rangle$ is an integer multiple of k_α for all $l \in L$.

The condition of reflectability ensures that a random walk cannot escape the fundamental chamber $\mathcal{C}$ without first hitting a wall of $\mathcal{C}$ [5].

For any positive integer r, and any $s, t \in L$, define $K : L \times L \to \mathbb{R}$ by

$$K_r(s,t) = WALK_r(s,t),$$

the number of paths from s to t using exactly r steps within S, and unconstrained by a chamber. Then by a fundamental result of Gessel and Zeilberger [4], we deduce the following result.

3.2. PROPOSITION.. *The function K_r is totally positive with respect to W; that is, $D_W K(s,t) \geq 0$ for all $s, t \in L$.*

As with the results in Section 2, Proposition 3.2 is established by finding an explicit formula that makes clear the nonnegativity of $D_W K$ on $L \times L$. In [4], it is proved that $D_W K(s,t)$ is exactly the number of r-step paths from s to t that *always* stay within the interior of the fundamental Weyl chamber $\mathcal{C}$. This yields nonnegativity immediately, proving that K is totally positive with respect to W.

In closing this section, we note that the results in [4] are also established for affine root systems.

4. Concluding remarks

As noted at the outset, this article adopts the point of view that harmonic analysis and representation theory provide natural settings in which to frame

the concept of total positivity. Indeed, nonnegative alternating sums that arise naturally in representation theory may be interpreted as establishing the total positivity of particular functions. For example, Weyl's character formula and Kostant's partition function [14] are both alternating sums in the sense of (4). In both cases the total positivity of the underlying function is immediate from the representation theory.

REFERENCES

1. W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. 1, 2nd. ed., Wiley, New York, 1957.
2. F. R. Gantmacher and M. G. Krein, *Sur le matrices complétement non-négatives et oscillatoires*, Compositio Math. **4** (1937), 445 – 476.
3. ———, *Oscillatory Matrices and Kernels and Small Vibrations of Mechanical Systems*, 2nd. ed., Moscow (Russian), 1950.
4. I. Gessel and D. Zeilberger, *Random walk in a Weyl chamber*, Proc. Amer. Math. Soc. **115** (1992), 27 – 31.
5. D. Grabiner and P. Magyar, *Random walks in Weyl chambers and the decomposition of tensor powers*, J. Algebraic Comb. **2** (1993), 239 – 260.
6. K. I. Gross and D. St. P. Richards, *Total positivity, spherical series, and hypergeometric functions of matrix argument*, J. Approximation Theory **59** (1989), 224 – 246.
7. ———, *Hypergeometric functions on complex matrix space*, Bull. Amer. Math. Soc. **24** (1991), 349 – 355.
8. ———, *Total positivity, finite reflection groups, and a formula of Harish-Chandra*, J. Approximation Theory (1995), to appear.
9. L. C. Grove and C. T. Benson, *Finite Reflection Groups*, 2nd. ed., Springer, New York, 1985.
10. Harish-Chandra, *Differential operators on a semisimple Lie algebra*, Amer. J. Math. **79** (1957), 87 – 120 = *Collected Works*, Vol. 2, (V. S. Varadarajan, ed.), Springer, New York.
11. K. D. Johnson, *A constructive approach to tensor product decompositions*, J. reine Angew. Math. **388** (1988), 129 – 148.
12. S. Karlin, *Total Positivity*, Vol. 1, Stanford University Press, Stanford, CA, 1968.
13. S. Karlin and Y. Rinott, *A generalized Cauchy-Binet formula and applications to total positivity and majorization*, J. Multivariate Analysis **27**, 284 – 299.
14. A. W. Knapp, *Representation Theory of Semisimple Groups: An Overview Based on Examples*, Princeton University Press, Princeton, N.J., 1986.
15. E. L. Lehmann, *Theory of Point Estimation*, Wadsworth, Belmont, CA, 1991.
16. I. J. Schoenberg, *Über Variationsverminde lineare Transformationen*, Math. Zeit. **32** (1930), 321 – 328.
17. J. R. Stembridge, *Immanants of totally positive matrices are nonnegative*, Bull. London Math. Soc. **23** (1993), 422 – 428.
18. D. Zeilberger, *Andre's reflection proof generalized to the many-candidate ballot problem*, Discrete Math. **44** (1983), 325 – 326.
19. D. P. Zhelobenko, *Compact Lie Groups and their Representations*, Amer. Math. Soc., Providence, RI, 1973.

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Covariant Differential Operators and Derived Functor Modules

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Dedicated to Ray A. Kunze

ABSTRACT. Let K be maximal compact in $SU(p,q)$ and let $\mathfrak{g} = A_{p+q-1}$. Covariant differential operators are used to identify nontrivial unitary irreducible $(\mathfrak{g}, K)$ modules in the family $\mathcal{F} = \{\Gamma^i(N(\lambda)^*) : i \geq 0\}$ where Γ denotes the Zuckerman functor and $N(\lambda)^*$ denotes the conjugate dual of a generalized Verma module $N(\lambda)$ whose irreducible quotient is unitary and not $N(\lambda)$. The nontrivial module $\Gamma^I(N(\lambda)^*)$ in $\mathcal{F}$ is identified here as unitary and irreducible, has cohomological degree I below the middle dimension, and is of highest weight type. The covariant differential operators used to make this identification arise as differentials in a spectral sequence induced by a Bernstein-Gelfand-Gelfand resolution of $N(\lambda)$. A Bott-Borel-Weil result is obtained by producing a Weyl group element σ of length I such that the irreducible unitary module $\Gamma^I(N(\lambda)^*)$ has highest weight $\sigma(\lambda + \rho) - \rho$.

1. Introduction

Derived functor induction is a general method for constructing admissible representations of a real reductive group G. The method uses the derived functors Γ^i, $i \geq 0$, of the Zuckerman functor Γ to produce $(\mathfrak{g}, K)$ modules where K is a maximal compact subgroup of G and $\mathfrak{g}$ denotes the complexification of the Lie algebra $\mathfrak{g}_0$ of G. One begins with a $(\mathfrak{g}, M)$ module V where M is a subgroup of K and considers the family $\mathcal{F}(V) = \{\Gamma^i(V) : i \geq 0\}$ of modules. The problem is to identify which of the nonzero modules in $\mathcal{F}(V)$ are unitarizable and irreducible. This problem becomes more tractable if one begins with $(\mathfrak{g}, M)$ modules V that are well understood and whose known structure one can readily

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exploit. For this reason it is reasonable to take V to be a generalized Verma module or its irreducible quotient. This is the strategy taken in [9], [14] and [20] and is the strategy we employ here as well. However, our method for identifying irreducible unitary modules in $\mathcal{F}(V)$ differs from the splitting criteria methods found in [9] and [14]. In this article we utilize covariant differential operators to make the desired identifications. The differential operator approach taken here allows us to prove a Bott-Borel-Weil result in which the derived functors Γ^i yield irreducible unitary highest weight $(\mathfrak{g}, K)$ modules where K is a maximal compact subgroup of $SU(p,q)$ and $\mathfrak{g} \cong A_{p+q-1}$. We develop some notation and more fully describe our methods and results.

Let $p, q \geq 1$ be integers and set $n = p + q$. Let $G = SU(p,q)$ and let G' denote either $SU(p+1, q-1)$ or $SU(p-1, q+1)$. Let K and L denote standard maximal compact subgroups of G and G', respectively. Set $M = L \cap K$. Let $\mathfrak{g}_0$ denote the Lie algebra of G and let $\mathfrak{g}$ denote the complexification of $\mathfrak{g}_0$. We make the standard choices for Cartan subalgebra $\mathfrak{t}$ and set $\{\alpha_i : 1 \leq i \leq n-1\}$ of simple roots. Let ω_i denote the fundamental dominant weight corresponding to the simple root α_i. If $G' = SU(p+1, q-1)$, let $\mathfrak{q}$ be the parabolic determined by ω_{p+1} and if $G' = SU(p-1, q+1)$, let $\mathfrak{q}$ be the parabolic determined by ω_{p-1}. Let S denote the middle dimension determined by $\mathfrak{q}$ and K. Let $F_L(\lambda)$ be an irreducible L module with highest weight λ and set $N(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} F_L(\lambda)$. We let $N(\lambda)^*$ denote the conjugate dual of $N(\lambda)$ and let $L(\lambda)$ denote the irreducible quotient of $N(\lambda)$.

Let $\Lambda(G')$ be the set of highest weights for unitary highest weight representations of G'. The set $\Lambda(G')$ was completely determined in [8]. Define the subset $\Lambda_r(G')$ of reduction points by $\Lambda_r(G') = \{\lambda \in \Lambda(G') : N(\lambda) \text{ is reducible}\}$. For $\lambda \in \Lambda_r(G')$, consider the family $\mathcal{F}(\lambda) = \{\Gamma^i(N(\lambda)^*) : i \geq 0\}$ of $(\mathfrak{g}, K)$ modules. We study the family $\mathcal{F}(\lambda)$ using a spectral sequence induced by a Bernstein-Gelfand-Gelfand (BGG) resolution of $N(\lambda)^*$ (cf. section four). The E_2 terms of the spectral sequence are computed in section five. From these terms one sees that for a certain index I the module $\Gamma^I(N(\lambda)^*)$ can be completely determined by the resulting E_∞ terms provided $I < S$. Specifically, there exist integers J and N such that $J + N = I$ and $E_\infty^{J,N} = \Gamma^I(N(\lambda)^*)$ if $I < S$. The integers J and N depend only on the cone in which λ lies. We refer to section six of [5] for the description of the cone decomposition of $\Lambda_r(G')$. We next show that $E_\infty^{J,N}$ arises as the kernel of a differential $d_r^{J,N}$ of the spectral sequence. This differential $d_r^{J,N}$ is related to a $(\mathfrak{g}, K)$ invariant differential operator called a covariant differential operator (CDO). Since CDOs are central to our method we now give a brief description of these operators and indicate how they are used.

Let (G, K) denote a Hermitian symmetric pair and let U_τ and U_μ denote holomorphically induced representations of G. A CDO $D_{\tau,\mu} : U_\tau \to U_\mu$ is a continuous intertwining map. We refer to [6] for the explicit form of $D_{\tau,\mu}$ in standard realizations of holomorphically induced representations. A certain distinguished class of nonzero CDOs has the special property that the space of K

finite vectors in $\mathrm{Ker}(D_{\tau,\mu})$ is an irreducible unitary highest weight $(\mathfrak{g}, K)$ module (cf. (3.2) [6]).

With this distinguished class of CDOs in mind, we outline our strategy. Since the space of such intertwining maps is one-dimensional (cf. [2], [3]) the operator $D_{\tau,\mu}$ is completely determined by the pair (τ, μ). In addition, all pairs of weights (τ, μ) for which $\mathrm{Ker}(D_{\tau,\mu})$ is irreducible and unitary are explicitly listed in [5]. We can therefore recognize a differential in a spectral sequence as (a multiple of) a distinguished CDO $D_{\tau,\mu}$ by identifying the appropriate parameters (τ, μ). Moreover, if it can be shown that a nonzero CDO appears, then the irreducibility and unitarizability of the kernel follow immediately.

In section six we identify the distinguished CDO pair (τ, μ) for which $d_r^{J,N} = cD_{\tau,\mu}, c \in \mathbb{C}$, and give the associated cone data. In particular, the dependency of the pair (τ, μ) on the form of the weight λ is made explicit. We view this dependence as relevant because it measures, to some extent, the manner in which CDOs factor through the induction process. After all, the assumption $\lambda \in \Lambda_r(G')$ implies that there exists a distinguished CDO $D_{\lambda,\mu(\lambda)}$ whose kernel in $N(\lambda)^*$ is an irreducible unitary $(\mathfrak{g}, L)$ module. The transfer of unitary structure by cohomological induction is reflected in the CDOs that appear as input and output.

The strategy outlined above is efficacious provided one can show that the above differential is nonzero. To this end, define the subset $\Lambda_r^M(G')$ of $\Lambda_r(G')$ by the condition that the irreducible L module $F_L(\lambda)$ remains irreducible upon restriction to the subgroup M. We show in section seven that for $\lambda \in \Lambda_r^M(G')$ the differential $d_r^{J,N}$ is nonzero. By the special property of these distinguished CDOs we conclude $\Gamma^I(N(\lambda)^*)$ is an irreducible unitary highest weight $(\mathfrak{g}, K)$ module with highest weight τ. The Bott-Borel-Weil result mentioned above is established in section five where it is shown that there exists a Weyl group element σ with $\ell(\sigma) = I$ and $\tau = \sigma(\lambda + \rho) - \rho$.

Explicit computations for the case $G' = SU(p+1, q-1)$ are presented in the body of the paper and analogous results for $G' = SU(p-1, q+1)$ are left to the final section. We now turn to some results in the literature related to this article.

In [14] more general Hermitian symmetric pairs (G', L) and (G, K) are studied and it is shown that for certain $\lambda \in \Lambda_r(G')$ the family $\{\Gamma^i(L(\lambda)) : i \geq 0\}$ contains irreducible unitary $(\mathfrak{g}, K)$ modules that are not highest weight modules. However, in order to satisfy a certain splitting criterion, λ is taken as the highest weight of a one-dimensional module $F_L(\lambda)$. Related results are obtained in [20] in a more general setting where (G, K) is no longer Hermitian symmetric but where the Cartan involutions θ and θ' corresponding to real forms G and G' commute. Certain unitary derived functor modules are explicitly computed in terms of their K types. These modules are examples of nonhighest weight modules and the condition $\dim(F_L(\lambda)) = 1$ is imposed to identify the K structure. In contrast, we produce unitary highest weight modules and there is no constraint on the

dimension of $F_L(\lambda)$.

Spectral sequences are a common tool in the computation of sheaf cohomology. For example, spectral sequences where used in [7] to show that solutions spaces of linear massless free-field equations can be realized as cohomology on twistor space. Generalizing this setting, the Penrose correspondence views sheaf cohomology on one flag manifold in terms of kernels and cokernels of invariant differential operators on sections of homogeneous vector bundles on a second flag manifold. We refer the reader to [1] for explicit details as well as many illustrative examples. The role of the Bernstein-Gelfand-Gelfand resolution and the resulting spectral sequence stands out in [1] as a dominant feature of this correspondence.

With these observations in mind, one can view aspects of what we do here as an algebraic version of the Penrose correspondence. Sheaf cohomology in the Penrose correspondence is replaced here by derived functor modules. The connection between analytic cohomology and the Zuckerman functors is well-known and is discussed in chapter eleven of [1]. What is new here is the role that covariant differential operators play in determining the irreducibility and unitarity of the cohomologically induced modules.

2. Preliminaries

Let p and q be positive integers and let $I_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$. Setting $n = p + q$, we put $SU(p,q) = \{g \in GL(n,\mathbb{C}) : g^* I_{p,q} g = I_{p,q}\}$ where * denotes conjugate transpose. We choose as maximal compact subgroup K of $SU(p,q)$ the subgroup

$$K = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} : A \in U(p), B \in U(q), \det(A) \cdot \det(B) = 1 \right\}.$$

Let $\theta(X) = -X^*$ where $X \in \mathbb{C}^{n \times n}$. The Lie algebra $\mathfrak{g}_0$ of $SU(p,q)$ decomposes according to the eigenvalues of θ. Specifically, the subalgebras

$$\mathfrak{k}_0 = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a \in \mathbb{C}^{p \times p}, b \in \mathbb{C}^{q \times q}, a^* = -a, b^* = -b, \mathrm{tr}(a) = -\mathrm{tr}(b) \right\},$$

$$\mathfrak{p}_0 = \left\{ \begin{bmatrix} 0 & c \\ c^* & 0 \end{bmatrix} : c \in \mathbb{C}^{p \times q} \right\}$$

correspond to the 1 and -1 eigenspaces, respectively. We drop the subscript "0" to denote complexification of a Lie algebra. Complexifying θ, we obtain the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g} \cong A_{n-1}$ where

$$\mathfrak{k} = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a \in \mathbb{C}^{p \times p}, b \in \mathbb{C}^{q \times q}, \mathrm{tr}(a) = -\mathrm{tr}(b) \right\}$$

and

$$\mathfrak{p} = \left\{ \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix} : c \in \mathbb{C}^{p \times q}, d \in \mathbb{C}^{q \times p} \right\}.$$

We also define subalgebras $\mathfrak{p}_+ = \left\{ \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} : x \in \mathbb{C}^{p\times q} \right\}$ and $\mathfrak{p}_- = \bar{\mathfrak{p}}_+$ where bar denotes conjugation in $\mathfrak{g}$ with respect to $\mathfrak{g}_0$.

Let $\mathfrak{t}_0$ denote the diagonal traceless matrices in $\mathfrak{g}_0$. Then $\mathfrak{t}$ is the subalgebra of diagonal traceless matrices in $\mathfrak{g}$ and the corresponding set of roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$ is $\Delta = \{e_i - e_j : i \neq j\}$ where $e_i \in \mathfrak{t}^*$ selects the i^{th} entry of the diagonal. We take $\{\alpha_i = e_i - e_{i+1} : 1 \leq i \leq n-1\}$ as the set of simple roots. The corresponding set of positive roots is $\Delta^+ = \{e_i - e_j : i < j\}$ and we let $\{\omega_i : 1 \leq i \leq n-1\}$ denote the associated set of fundamental dominant weights.

A linear functional $\tau \in \mathfrak{t}^*$ that takes purely imaginary values on $\mathfrak{t}_0$ is called a weight. We follow (5.2.1) [18] and define parabolic subalgebras $\mathfrak{q} = \mathfrak{q}(\omega)$ of $\mathfrak{g}$ for each weight ω. Let $\mathfrak{g}_\alpha$ denote the root space corresponding to $\alpha \in \Delta$. Define subsets of Δ by

$$\begin{aligned} \Delta(\ell) &= \{\alpha \in \Delta : (\alpha, \omega) = 0\}, \\ \Delta(\mathfrak{u}) &= \{\alpha \in \Delta : (\alpha, \omega) > 0\}, \\ \Delta(\mathfrak{q}) &= \{\alpha \in \Delta : (\alpha, \omega) \geq 0\}, \end{aligned}$$

and corresponding subalgebras by

$$\ell = \mathfrak{t} \oplus \Big(\bigoplus_{\alpha\in\Delta(\ell)} \mathfrak{g}_\alpha \Big), \quad \mathfrak{u} = \bigoplus_{\alpha\in\Delta(\mathfrak{u})} \mathfrak{g}_\alpha, \quad \mathfrak{q} = \ell \oplus \mathfrak{u}.$$

Then $\mathfrak{q}$ is a parabolic subalgebra satisfying $\theta(\ell) = \ell$, $\theta(\mathfrak{u}) = \mathfrak{u}$, $\bar{\ell} = \ell$ and $\mathfrak{q} = (\mathfrak{q} \cap \mathfrak{k}) \oplus (\mathfrak{q} \cap \mathfrak{p})$. The decomposition $\mathfrak{q} = \ell \oplus \mathfrak{u}$ is called the θ stable Levi decomposition. Note that the parabolic $\mathfrak{q}(\omega_i)$ is defined for each fundamental dominant weight ω_i, $1 \leq i \leq n-1$. In particular, we observe that $\mathfrak{k} \oplus \mathfrak{p}_+$ is the θ stable Levi decomposition of $\mathfrak{q}(\omega_p)$.

In this article we will be concerned with the two parabolics $\mathfrak{q}(\omega_{p-1})$ and $\mathfrak{q}(\omega_{p+1})$. In order that these be defined, we henceforth assume that either $p \geq 2$ or $q \geq 2$. In the sections that follow, we shall give explicit calculations using only the parabolic $\mathfrak{q}(\omega_{p+1})$ and reserve the notation ℓ and $\mathfrak{u}$ for this particular parabolic. The results for the parabolic $\mathfrak{q}(\omega_{p-1})$ are similar and we state these results in section eight, leaving the omitted details to the reader.

For later reference we explicitly determine the parabolic subalgebra $\mathfrak{q} = \mathfrak{q}(\omega_{p+1})$. We clearly have $\Delta(\ell) = \{\pm(e_i - e_j) : p+1 < i < j \leq n \text{ or } 1 \leq i < j \leq p+1\}$ and $\Delta(\mathfrak{u}) = \{e_i - e_j : 1 \leq i \leq p+1 \text{ and } p+2 \leq j \leq n\}$ with corresponding algebras:

$$\begin{aligned} \ell &= \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : \operatorname{tr}(a) = -\operatorname{tr}(b), a \in \mathbb{C}^{(p+1)\times(p+1)}, b \in \mathbb{C}^{(q-1)\times(q-1)} \right\}, \\ \mathfrak{u} &= \left\{ \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} : x \in \mathbb{C}^{(p+1)\times(q-1)} \right\}. \end{aligned}$$

Let $Z(\ell)$ denote the center of ℓ and set $\ell^{ss} = [\ell, \ell]$. Then $\ell = \ell^{ss} \oplus Z(\ell)$ as the

direct sum of ideals. Specifically, we have

$$\ell^{ss} = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : \operatorname{tr}(a) = \operatorname{tr}(b) = 0, a \in \mathbb{C}^{(p+1)\times(p+1)}, b \in \mathbb{C}^{(q-1)\times(q-1)} \right\},$$

$$Z(\ell) = \left\{ c \begin{bmatrix} -(q-1)I_{p+1} & 0 \\ 0 & (p+1)I_{q-1} \end{bmatrix} : c \in \mathbb{C} \right\},$$

so that $Z(\ell) \subset \mathfrak{t}$, $\dim(Z(\ell)) = 1$ and $\ell^{ss} \cong A_p \oplus A_{q-2}$ is semisimple.

We now consider the parabolic $\mathfrak{q}_1 = \mathfrak{q}(\omega_p + \omega_{p+1})$ and denote the Levi decomposition as $\mathfrak{q}_1 = \mathfrak{m} \oplus \mathfrak{u}_1$. It is clear that $\Delta(\mathfrak{m}) = \Delta(\ell) \cap \Delta(\mathfrak{k})$ and $\Delta(\mathfrak{u}_1) = \Delta(\mathfrak{u}) \cup \Delta(\mathfrak{p}_+)$ so that $\mathfrak{m} = \ell \cap \mathfrak{k}$, $\mathfrak{u}_1 = \mathfrak{u} + \mathfrak{p}_+$ and $\mathfrak{q}_1 = \mathfrak{q}(\omega_{p+1}) \cap \mathfrak{q}(\omega_p)$. Set $\mathfrak{m}^{ss} = [\mathfrak{m}, \mathfrak{m}]$ and let $Z(\mathfrak{m})$ denote the center of $\mathfrak{m}$. Since $\mathfrak{m}$ has the block form

$$\mathfrak{m} = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} : a \in \mathbb{C}^{p\times p}, c \in \mathbb{C}^{(q-1)\times(q-1)}, -b = \operatorname{tr}(a) + \operatorname{tr}(c) \right\},$$

we find that the decomposition $\mathfrak{m} = \mathfrak{m}^{ss} \oplus Z(\mathfrak{m})$ takes the form

$$\mathfrak{m}^{ss} = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{bmatrix} : a \in \mathbb{C}^{p\times p}, c \in \mathbb{C}^{(q-1)\times(q-1)}, \operatorname{tr}(a) = \operatorname{tr}(c) = 0 \right\},$$

$$Z(\mathfrak{m}) = \left\{ \begin{bmatrix} aI_p & 0 & 0 \\ 0 & -(pa + (q-1)b) & 0 \\ 0 & 0 & bI_{q-1} \end{bmatrix} : a, b \in \mathbb{C} \right\}.$$

In particular, we see that $Z(\ell) \subset Z(\mathfrak{m}) \subset \mathfrak{t}$, $\dim(Z(\mathfrak{m})) = 2$ and $\mathfrak{m}^{ss} \cong A_{p-1} \oplus A_{q-2}$ is semisimple. Note that the simple system of roots $\mathcal{S}(\mathfrak{m})$ for the set of positive roots $\Delta^+(\mathfrak{m}) = \Delta^+ \cap \Delta(\mathfrak{m})$ of $\mathfrak{m}$ is $\mathcal{S}(\mathfrak{m}) = \{\alpha_i : i \neq p, p+1\}$.

Let L denote the maximal compact subgroup of $SU(p+1, q-1)$ given by

$$L = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} : A \in U(p+1), B \in U(q-1), \det(A) \cdot \det(B) = 1 \right\}. \tag{2.1}$$

It is clear that the complexification of the Lie algebra of L is ℓ. We set $M = L \cap K$. Then

$$M = \left\{ \begin{bmatrix} A & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & B \end{bmatrix} : A \in U(p), B \in U(q-1), c \in \mathbb{C}, c \cdot \det(A) \cdot \det(B) = 1 \right\}. \tag{2.2}$$

In particular, note that $M \cong U(p) \times U(q)$ is connected and that the complexified Lie algebra of M is $\mathfrak{m}$.

3. Spectral Sequences

We briefly describe some standard results in homological algebra which provide the basic tools to compute cohomology. Details may be found in chapter

fifteen of [4] or chapter eleven of [16]. The goal of this section is to use an appropriate resolution of a complex to construct a spectral sequence which converges to the desired derived functor module. We begin with a definition.

A proper injective resolution M of a complex $N : 0 \to N^0 \to N^1 \to \cdots$ is a double complex $M = \{M^{r,s} : r, s \geq 0\}$ of injective modules with commuting diagram

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & M^{0,1} & \to & M^{1,1} & \to & M^{2,1} & \to & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & M^{0,0} & \to & M^{1,0} & \to & M^{2,0} & \to & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & N^0 & \to & N^1 & \to & N^2 & \to & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{3.1}
$$

such that, for each $m \geq 0$, the m^{th} column $M^{m,*}$ of M is a deleted injective resolution of N^m and each row in (3.1) is a complex. Furthermore, if $M^{*,m}$ denotes the m^{th} row and if

$$
\begin{aligned}
Z^{n,m} &= \operatorname{Ker} : M^{n,m} \xrightarrow{d^{n,m}} M^{n+1,m}, \\
B^{n,m} &= \operatorname{im} : M^{n-1,m} \xrightarrow{d^{n-1,m}} M^{n,m}, \\
H^{n,m} &= Z^{n,m}/B^{n,m}
\end{aligned}
$$

denote the cocycles, coboundaries and cohomology of the m^{th} row, then the following complexes are injective resolutions:

$$
\begin{array}{ccccccccc}
0 & \to & Z^n(N) & \to & Z^{n,0} & \to & Z^{n,1} & \to & \cdots \\
0 & \to & B^n(N) & \to & B^{n,0} & \to & B^{n,1} & \to & \cdots \\
0 & \to & H^n(N) & \to & H^{n,0} & \to & H^{n,1} & \to & \cdots .
\end{array}
$$

By the dual of Lemma 11.33 [16], if $\mathcal{C}$ is a category of modules having enough injectives and if N is a complex, then N has a proper injective resolution.

(3.2) PROPOSITION. *Let Γ denote a covariant left exact additive functor defined on a category $\mathcal{C}$ of modules having enough injectives. Let A be a module in $\mathcal{C}$ and suppose*

$$
0 \to A \to N^0 \to N^1 \to \cdots \tag{3.3}
$$

is a resolution of A in the category $\mathcal{C}$. Then there exists a first quadrant spectral sequence with

$$
E_2^{n,m} = H^n(\Gamma^m(N)) \underset{n}{\Longrightarrow} \Gamma^{n+m}(A)
$$

where N denotes the deleted complex $0 \to N^0 \to N^1 \to \cdots$. The differential $d_r : E_r \to E_r$ has bidegree $(r, 1-r)$.

PROOF. Let M be a proper injective resolution of N. The exact sequence

$$0 \to Z^{n,m} \to M^{n,m} \xrightarrow{d^{n,m}} B^{n+1,m} \to 0$$

splits since $Z^{n,m}$ is injective. Likewise, the exact sequence $0 \to B^{n,m} \to Z^{n,m} \to H^{n,m} \to 0$ splits since $B^{n,m}$ is injective. Since the functor Γ is additive, it preserves split exact sequences. From the split exact sequence

$$0 \to \Gamma(Z^{n,m}) \to \Gamma(M^{n,m}) \xrightarrow{\Gamma(d^{n,m})} \Gamma(B^{n+1,m}) \to 0,$$

we have $\mathrm{Ker}(\Gamma(d^{n,m})) \cong \Gamma(Z^{n,m})$ and $\mathrm{im}(\Gamma(d^{n,m})) = \Gamma(B^{n+1,m})$. Consequently, $\mathrm{Ker}(\Gamma(d^{n,m}))/\mathrm{im}(\Gamma(d^{n-1,m})) \cong \Gamma(Z^{n,m})/\Gamma(B^{n,m})$. From the split exact sequence

$$0 \to \Gamma(B^{n,m}) \to \Gamma(Z^{n,m}) \to \Gamma(H^{n,m}) \to 0$$

we have $\Gamma(Z^{n,m})/\Gamma(B^{n,m}) \cong \Gamma(H^{n,m})$. But then one has

$$H^n(\Gamma(M^{*,m})) = \mathrm{Ker}(\Gamma(d^{n,m}))/\mathrm{im}(\Gamma(d^{n-1,m})) \cong \Gamma(H^{n,m}). \tag{3.4}$$

We now consider the double complex $\Gamma(M) = \{\Gamma(M^{r,s}) : r, s \geq 0\}$ and cohomology of the associated total complex $T(\Gamma(M))$. We begin by determining the E_2 terms of the spectral sequences arising from the first and second filtrations of $T(\Gamma(M))$.

The E_2 terms ${}'E_2^{n,m}$ arising from the first filtration are obtained by initially taking cohomology up the columns of $\Gamma(M)$, obtaining a double complex whose m^{th} row has the form

$$0 \to \Gamma^m(N^0) \to \Gamma^m(N^1) \to \Gamma^m(N^2) \to \cdots$$

since the columns of M form injective resolutions. One then takes cohomology down the rows to obtain ${}'E_2^{n,m} = H^n(\Gamma^m(N))$.

The E_2 terms ${}''E_2^{n,m}$ of the spectral sequence arising from the second filtration are obtained by initially taking cohomology down the rows of $\Gamma(M)$ thereby obtaining a double complex whose n^{th} column is of the form

$$0 \to H^n(\Gamma(M^{*,0})) \to H^n(\Gamma(M^{*,1})) \to \cdots$$

and then taking cohomology up the columns. However, by (3.4), the n^{th} column is

$$\Gamma(H^{n,*}) : 0 \to \Gamma(H^{n,0}) \to \Gamma(H^{n,1}) \to \cdots$$

Since the modules $H^{n,j}$ form an injective resolution of $H^n(N)$, we have $H^m(\Gamma(H^{n,*})) = \Gamma^m(H^n(N))$ by definition of the derived functor Γ^m of Γ. But (3.3) is a resolution of A and so is exact. Therefore, we have

$$\Gamma^m(H^n(N)) = \begin{cases} \Gamma^m(H^0(N)) \cong \Gamma^m(A) & \text{if } n = 0 \\ 0 & \text{if } n > 0. \end{cases}$$

Since ${}''E_2^{m,n} = \Gamma^m(H^n(N)) = 0$ for $n > 0$, the second spectral sequence collapses and we have

$$H^m(T(\Gamma(M))) \cong {}''E_2^{m,0} = \Gamma^m(A).$$

It now follows that

$$'E_2^{n,m} = H^n(\Gamma^m(N)) \underset{n}{\Longrightarrow} H^{n+m}(T(\Gamma(M))) = \Gamma^{n+m}(A).$$

Moreover, the differential d_r has bidegree $(r, 1-r)$. This completes the proof.

4. A BGG resolution

The goal of this section is to choose a convenient $(\mathfrak{g}, M)$ module A and construct a suitable resolution of A by $(\mathfrak{g}, M)$ modules in order to apply (3.2). We refer the reader to section 6.1 of [18] for the precise definition of the category $\mathcal{C}(\mathfrak{g}, M)$ of $(\mathfrak{g}, M)$ modules and for basic properties that this category enjoys. In sections four through seven we assume $q > 1$ and work entirely with the parabolic $\mathfrak{q} = \mathfrak{q}(\omega_{p+1})$, leaving the results for the parabolic $\mathfrak{q}(\omega_{p-1})$ to section eight.

The construction of our desired resolution requires several steps which we now outline. We begin with a representation space $F_L(\lambda)$ for an irreducible representation of the compact group L in (2.1) with highest weight λ. We first apply the construction in [13] to the semisimple Lie algebra ℓ^{ss} in order to obtain a resolution of $F_L(\lambda)$ by ℓ^{ss} modules. We then show that this resolution extends to a resolution of $F_L(\lambda)$ by (ℓ, M) modules. Finally, by applying appropriate exact functors, we obtain a resolution of the conjugate dual of a generalized Verma module by $(\mathfrak{g}, M)$ modules.

Following [13], we begin with a choice of a subset $\mathcal{S}$ of simple roots of the simple roots of ℓ^{ss}. This choice determines the parabolic subalgebra $\mathcal{P}_{\mathcal{S}}$ of ℓ^{ss} from which we construct generalized Verma modules on which ℓ^{ss} acts. We take $\mathcal{S}$ to be the subset $\mathcal{S}(\mathfrak{m}) = \{\alpha_i : i \neq p, p+1\}$ of simple roots of $\mathfrak{m}$. Then the simple Lie algebra generated by $\mathfrak{g}^{\pm\alpha}$, $\alpha \in \mathcal{S}(\mathfrak{m})$, is $\mathfrak{m}^{ss}$ given in section two. We set

$$\mathfrak{u}_{\mathcal{S}} = \bigoplus_{\alpha \in \Delta^+(\ell) \setminus \Delta(\mathfrak{m})} \mathfrak{g}_\alpha, \quad \mathfrak{r}_{\mathcal{S}} = \mathfrak{m}^{ss} + (\mathfrak{t} \cap \ell^{ss}) \text{ and } \mathcal{P}_{\mathcal{S}} = \mathfrak{r}_{\mathcal{S}} \oplus \mathfrak{u}_{\mathcal{S}}.$$

Then from [13] one knows that $\mathfrak{r}_{\mathcal{S}}$ is a reductive subalgebra of ℓ^{ss} and $\mathcal{P}_{\mathcal{S}}$ is parabolic subalgebra of ℓ^{ss}. We may see this directly by noting that $\mathfrak{t} \cap \ell^{ss} = \operatorname{span}_{\mathbb{C}}\{h_i : i \neq p+1\}$ where h_i denotes the diagonal matrix with 1 in the i^{th} entry and -1 in the $(i+1)^{\text{st}}$ entry. Consequently $\mathfrak{r}_{\mathcal{S}} = \mathfrak{m}^{ss} \oplus \mathbb{C} h_p$ is of the block diagonal form

$$\mathfrak{r}_{\mathcal{S}} = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} : a \in \mathbb{C}^{p \times p}, c \in \mathbb{C}^{(q-1)\times(q-1)}, \operatorname{tr}(a) = -b, \operatorname{tr}(c) = 0 \right\}.$$

The subalgebra $\mathfrak{r}_{\mathcal{S}}$ has a one-dimensional center $Z(\mathfrak{r}_{\mathcal{S}})$ spanned by

$$z_0 = \begin{bmatrix} I_p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

One has $Z(\mathfrak{r}_S) \subset Z(\mathfrak{m})$ and $\mathfrak{r}_S = \mathfrak{m}^{ss} \oplus \mathbb{C}z_0$. In particular, $\mathfrak{r}_S$ is clearly reductive. Set $\Delta(\mathfrak{u}_S) = \Delta^+(\ell) \setminus \Delta(\mathfrak{m})$. Then clearly $\Delta(\mathfrak{u}_S) = \{\alpha_j + \cdots + \alpha_p : 1 \leq j \leq p\}$ and $\mathfrak{u}_S$ has the explicit form

$$\mathfrak{u}_S = \left\{ \begin{bmatrix} 0 & \cdots & 0 & x_{1,p+1} & 0 & \cdots & 0 \\ & & & \vdots & & & \\ & & & x_{p,p+1} & & & \\ \vdots & & & 0 & & & \vdots \\ & & & \vdots & & & \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} : x_{j,p+1} \in \mathbb{C}, 1 \leq j \leq p \right\}. \tag{4.1}$$

From (4.1) one has $\mathfrak{u}_S = \ell \cap \mathfrak{p}_+$.

Recall that a weight τ is said to be $\Delta^+(\mathfrak{m})$ dominant (resp. $\mathfrak{m}$ integral) if $2(\tau, \alpha)/(\alpha, \alpha)$ is nonnegative (resp. an integer) for all $\alpha \in \Delta^+(\mathfrak{m})$. For such a $\Delta^+(\mathfrak{m})$ dominant, $\mathfrak{m}$ integral weight $\tau \in \mathfrak{t}^*$, there exists an irreducible $\mathfrak{m}^{ss}$ module $F_S(\tau)$ with highest weight τ. Define a $\mathcal{P}_S$ action on $F_S(\tau)$ by letting $\mathfrak{u}_S$ act by 0 and letting the central element z_0 of $\mathfrak{r}_S$ act by $\tau(z_0)$. Define an ℓ^{ss} module $N_S(\tau)$ by

$$N_S(\tau) = U(\ell^{ss}) \otimes_{U(\mathcal{P}_S)} F_S(\tau). \tag{4.2}$$

Let $\{\sigma_i = \sigma_{\alpha_i}, 1 \leq i \leq n-1\}$ denote the set of simple reflections and let $\mathcal{W}$ denote the Weyl group of $\mathfrak{g}$. Let $\mathcal{W}_\ell$ (resp. $\mathcal{W}_S$) denote the subgroup generated by $\{\sigma_i : i \neq p+1\}$ (resp. $\{\sigma_i : i \neq p, p+1\}$). We set

$$\mathcal{W}_S^1 = \{\omega \in \mathcal{W}_\ell : \Delta^+(\mathfrak{m}) \subset \omega\Delta^+(\ell)\}. \tag{4.3}$$

For $\omega \in \mathcal{W}$, we set $\Delta^+(\omega) = \{\alpha \in \Delta^+ : \omega^{-1}\alpha \notin \Delta^+\}$ and define the length of $\omega, \ell(\omega)$, to be the number of elements in $\Delta^+(\omega)$. Let ρ_ℓ denote half the sum of the roots in $\Delta^+(\ell)$.

Let the weight λ be $\Delta^+(\ell)$ dominant integral, i.e. $2(\lambda, \alpha)/(\alpha, \alpha)$ is a nonnegative integer for all $\alpha \in \Delta^+(\ell)$. It is well-known that $\omega(\lambda)$ is $\Delta^+(\mathfrak{m})$ dominant integral for all $\omega \in \mathcal{W}_S^1$. Indeed, sending $\Delta^+(\ell)$ dominant weights to $\Delta^+(\mathfrak{m})$ dominant weights is a property that completely characterizes the set $\mathcal{W}_S^1$ (cf. (3.2.2) [18]). It follows that the module $N_S(\omega(\lambda + \rho_\ell) - \rho_\ell)$ is defined for all $\omega \in \mathcal{W}_S^1$.

We now state Theorem 4.3 [13] as it applies to the semisimple Lie algebra ℓ^{ss} and parabolic $\mathcal{P}_S$. Note that from (4.1) we have $\dim(\mathfrak{u}_S) = p$.

(4.4) THEOREM [13]. *Let λ be an $\Delta^+(\ell)$ dominant integral weight and let $F_{\ell^{ss}}(\lambda)$ denote the irreducible ℓ^{ss} module with highest weight λ. Set*

$$C_j = \coprod_{\omega \in W_s^1, \ell(\omega) = j} N_S(\omega(\lambda + \rho_\ell) - \rho_\ell), \quad \text{for } 0 \leq j \leq p.$$

Then there exist ℓ^{ss} module maps $d_j, 1 \leq j \leq p$, and ϵ such that the sequence

$$0 \to C_p \xrightarrow{d_p} \cdots \xrightarrow{d_1} C_0 \xrightarrow{\epsilon} F_{\ell^{ss}}(\lambda) \to 0 \tag{4.5}$$

is exact. Here ϵ is the canonical ℓ^{ss} module epimorphism.

We refer to the resolution in (4.5) as the Bernstein-Gelfand-Gelfand (BGG) resolution of $F_{\ell^{ss}}(\lambda)$. In our setting the exact sequence (4.5) simplifies. One readily computes $\mathcal{W}_S^1$ to be

$$\mathcal{W}_S^1 = \{1, \sigma_p, \sigma_p\sigma_{p-1}, \dots, \sigma_p \cdots \sigma_1\}.$$

This is most conveniently done by using the algorithm on page 41 of [1]. In particular note that $\mathrm{Card}(\mathcal{W}_S^1) = p+1$ and the elements of $\mathcal{W}_S^1$ are determined by their length. We then have

$$\begin{aligned} &C_0 = N_S(\lambda), \\ &C_j = N_S(\sigma_p \cdots \sigma_{p-j+1}(\lambda + \rho_\ell) - \rho_\ell)), \text{ for } 1 \le j \le p. \end{aligned} \tag{4.6}$$

It is the simple form of each C_j that motivated the choice of parabolics $\mathfrak{q}(\omega_{p+1})$ and $\mathfrak{q}(\omega_{p-1})$ in section two.

We now extend the sequence (4.5) to a sequence of (ℓ, M) modules. To do this we need to begin with analytically integral weights τ and corresponding representations of M. A linear functional $\tau \in \mathfrak{t}^*$ is said to be analytically integral if $\tau(H) \in 2\pi i\mathbb{Z}$ whenever $H \in \mathfrak{t}_0$ satisfies $\exp(H) = 1$. Clearly an analytically integral functional is a weight. Since $M \cong U(p) \times U(q-1)$, M is a compact linear connected reductive group. By (4.28) [11], if $\tau \in \mathfrak{t}^*$ is analytically integral and $\Delta^+(\mathfrak{m})$ dominant, then there exists an irreducible representation π_τ of M on $F_M(\tau)$ with highest weight τ. Now each $z \in Z(\mathfrak{m})$ acts by $\tau(z)$ on $F_M(\tau)$. Since the central element z_0 of $\mathfrak{r}_S$ lies in $Z(\mathfrak{m})$ it follows that $F_M(\tau)$ is an irreducible $\mathfrak{m}^{ss}$ module on which z_0 acts by $\tau(z_0)$. Consequently, the ℓ^{ss} module $N_S(\tau)$ in (4.2) is defined. We put an (ℓ, M) action on $N_S(\tau)$ by letting M act by $\mathrm{Ad} \otimes \pi_\tau$ and letting $Z(\ell)$ act by $\tau(z)$, $z \in Z(\ell)$.

We may alternately view the (ℓ, M) modules $N_S(\tau)$ as follows. Define subalgebras $\mathfrak{r}_\ell = \mathfrak{r}_S \oplus Z(\ell)$ and $\mathcal{P}_\ell = \mathfrak{r}_\ell \oplus \mathfrak{u}_S$. Since $Z(\ell) \oplus \mathbb{C}z_0 = Z(\mathfrak{m})$, we have $\mathfrak{r}_\ell = \mathfrak{m}$. Since $\mathfrak{m} = \ell \cap \mathfrak{k}$ and $\mathfrak{u}_S = \ell \cap \mathfrak{p}_+$, one has $\mathcal{P}_\ell = \ell \cap \mathfrak{q}(\omega_p)$ so that $\mathcal{P}_\ell$ is a parabolic subalgebra of ℓ. The space $F_M(\tau)$ is a $\mathcal{P}_\ell$ module if we again define $\mathfrak{u}_S$ to act trivially. Define an M action on $U(\ell) \otimes_{U(\mathcal{P}_\ell)} F_M(\tau)$ by $\mathrm{Ad} \otimes \pi_\tau$. It is then clear that as (ℓ, M) modules we have

$$N_S(\tau) \cong U(\ell) \otimes_{U(\mathcal{P}_\ell)} F_M(\tau). \tag{4.7}$$

(4.8) PROPOSITION. *Let λ be an analytically integral weight. Then for each $\sigma \in \mathcal{W}_\ell$, the weight $\sigma(\lambda + \rho_\ell) - \rho_\ell$ is analytically integral. Furthermore, if $z \in Z(\ell)$, then one has $(\sigma(\lambda + \rho_\ell) - \rho_\ell)(z) = \lambda(z)$.*

PROOF. The first statement follows by induction on $\ell(\sigma)$. One uses that $\lambda + \rho_\ell$ is ℓ integral and that each $\alpha \in \Delta(\ell)$ is analytically integral. We omit the details. Since $\dim(Z(\ell)) = 1$, it suffices to verify the second statement for a nonzero $z \in Z(\ell) \cap i\mathfrak{t}_0$. Normalize z so that $\tau(z) = (\tau, \omega_{p+1})$ for all weights τ. Since each element of $\mathcal{W}_\ell$ fixes ω_{p+1}, we have $(\sigma(\lambda + \rho_\ell) - \rho_\ell)(z) = (\sigma(\lambda + \rho_\ell) - \rho_\ell, \omega_{p+1}) = (\lambda + \rho_\ell, \sigma^{-1}(\omega_{p+1})) - (\rho_\ell, \omega_{p+1}) = (\lambda + \rho_\ell, \omega_{p+1}) - (\rho_\ell, \omega_{p+1}) = (\lambda, \omega_{p+1}) = \lambda(z)$.

(4.9) COROLLARY. *If $\lambda \in \mathfrak{t}^*$ is analytically integral and $\Delta^+(\ell)$ dominant, then for each $j \in \{1, \dots, p\}$ the weight $\lambda_j = \sigma_p \cdots \sigma_{p-j+1}(\lambda + \rho_\ell) - \rho_\ell$ is analytically integral and $\Delta^+(\mathfrak{m})$ dominant.*

PROOF. The weight λ_j is analytically integral by (4.8) and is $\Delta^+(\mathfrak{m})$ dominant since $\sigma_p \cdots \sigma_{p-j+1}$ lies in $\mathcal{W}_S^1$.

(4.10) PROPOSITION. *Let τ_1 and τ_2 be analytically integral and $\Delta^+(\mathfrak{m})$ dominant. Assume $d : U(\ell) \otimes_{U(\mathcal{P}_\ell)} F_M(\tau_1) \to U(\ell) \otimes_{U(\mathcal{P}_\ell)} F_M(\tau_2)$ is an ℓ module map. Then d is an M module map as well.*

PROOF. Let π_1 (resp. π_2) denote the representation of M on $F_M(\tau_1)$ (resp. $F_M(\tau_2)$). We need to show that

$$d(\mathrm{Ad}(g) \otimes \tau_1(g)(\ell_1 \cdots \ell_r \otimes v)) = \mathrm{Ad}(g) \otimes \tau_2(g) d(\ell_1 \cdots \ell_r \otimes v) \tag{4.11}$$

for $\ell_1, \dots, \ell_r \in \ell, v \in F_M(\tau_1)$ and $g \in M$. Since d is an ℓ map, the left side of (4.11) is $(\mathrm{Ad}(g)\ell_1) \cdots (\mathrm{Ad}(g)\ell_r) d(1 \otimes \pi_1(g)v)$ and the right side of (4.11) is $(\mathrm{Ad}(g)\ell_1) \cdots (\mathrm{Ad}(g)\ell_r)(\mathrm{Ad}(g) \otimes \pi_2(g)(d(1 \otimes v)))$. It therefore suffices to show that

$$d(1 \otimes \pi_1(g)v) = \mathrm{Ad}(g) \otimes \pi_2(g)(d(1 \otimes v)) \text{ for all } g \in M \text{ and } v \in F_M(\tau_1). \tag{4.12}$$

In order to prove (4.12) we set $V = 1 \otimes F_M(\tau_1)$ and let ϕ denote the restriction of d to V. We abuse notation and write π_1 for the action of M on V. Since $\mathrm{im}(\phi)$ is a finite-dimensional space consisting of M finite vectors, there exists a finite-dimensional M invariant subspace W containing $\mathrm{im}(\phi)$. Let $T(g)$ denote the restriction of $\mathrm{Ad}(g) \otimes \pi_2(g)$, $g \in M$, to W. Since d is $\mathfrak{m}$ equivariant, the space $\mathrm{im}(\phi)$ is $\mathfrak{m}$ invariant. Since M is connected and T is a finite-dimensional representation, it follows from Theorem 3.4 p. 356 and VI p. 511 [15] that $\mathrm{im}(\phi)$ is $T(M)$ invariant. Let $T_0(g)$, $g \in M$, denote the restriction of $T(g)$ to $\mathrm{im}(\phi)$. Since π_1 (resp. T_0) is a finite-dimensional representation, it is an analytic homomorphism of the Lie group M into the Lie group $GL(V)$ (resp. $GL(\mathrm{im}(\phi))$) by VI p. 511 [15]. By (3.4.25) p.354 [15] we know $\pi_1(\exp(x)) = \exp(d\pi_1(x))$ and $T_0(\exp(x)) = \exp(dT_0(x))$ for all x in the Lie algebra $\mathfrak{m}_0$ of M. Since $\phi d\pi_1(x) = dT_0(x)\phi$, we have $\phi\pi_1(\exp(x)) = T_0(\exp(x))\phi$ for $x \in \mathfrak{m}_0$. Since M is connected we conclude that $\phi\pi_1(g) = T_0(g)\phi$ for all $g \in M$. Thus ϕ intertwines the M actions. This proves (4.12) and consequently (4.11).

(4.13) PROPOSITION. *Let $\lambda \in \mathfrak{t}^*$ be analytically integral and $\Delta^+(\ell)$ dominant. Let $F_M(\lambda_j)$ denote the representation space for the irreducible representation of M with highest weight λ_j given in (4.9). Put*

$$C_j^{(\ell,M)} = U(\ell) \otimes_{U(\mathcal{P}_\ell)} F_M(\lambda_j) \text{ for } 1 \le j \le p \text{ and}$$
$$C_0^{(\ell,m)} = U(\ell) \otimes_{U(\mathcal{P}\ell)} F_M(\lambda).$$

There exist (ℓ, M) module maps $d_j, 1 \le j \le p$, and ϵ such that the sequence

$$0 \to C_p^{(\ell,M)} \xrightarrow{d_p} \cdots \xrightarrow{d_1} C_0^{(\ell,M)} \xrightarrow{\epsilon} F_L(\lambda) \to 0 \tag{4.14}$$

is exact. Here ϵ is the canonical (ℓ, M) module epimorphism.

PROOF. By (4.7), the (ℓ, M) modules $C_j^{(\ell,M)}$, $0 \leq j \leq p$, are just the ℓ^{ss} modules C_j in (4.6) on which an action of $Z(\ell)$ and an action of M have been defined. We need only show that the ℓ^{ss} module maps given in (4.4) automatically extend to (ℓ, M) module maps. We first observe that on $C_j^{(\ell,M)}$ the action of $z \in Z(\ell)$ is by the scalar $\lambda_j(z) = (\sigma_p \cdots \sigma_{p-j+1}(\lambda + \rho_\ell) - \rho_\ell)(z)$. Since $\sigma_p \cdots \sigma_{p-j+1} \in \mathcal{W}_\ell$, we have by (4.8) that $\lambda_j(z) = \lambda(z)$. Thus the action of $z \in Z(\ell)$ on each $C_j^{(\ell,M)}$, $0 \leq j \leq p$, is by the same scalar $\lambda(z)$. Since $\ell = \ell^{ss} \oplus Z(\ell)$ we conclude that each ℓ^{ss} map in (4.4) is an ℓ map. Finally, each ℓ^{ss} map in (4.4) is an M map by (4.10). This completes the proof.

Recall our choice of parabolic subalgebras $\mathfrak{q} = \ell \oplus \mathfrak{u}$ and $\mathfrak{q}_1 = \mathfrak{m} \oplus \mathfrak{u}_1$ of $\mathfrak{g}$ and the parabolic subalgebra $\mathcal{P}_\ell = \mathfrak{m} \oplus \mathfrak{u}_\mathcal{S}$ of ℓ. If W is an (ℓ, M) module, we define $\mathfrak{u}$ to act trivially on W and thus make W a $(\mathfrak{q}, M)$ module. We make $U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} W$ a $(\mathfrak{g}, M)$ module by letting M act by the tensor action $\mathrm{Ad} \otimes \pi_W$ where π_W is the representation of M on W. In particular, if π_λ is an irreducible representation of L on the space $F_L(\lambda)$ with highest weight λ, we set

$$N_\mathfrak{q}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} F_L(\lambda). \tag{4.15}$$

In a similar way, if V is a $(\mathfrak{m}, M)$ module we let $\mathfrak{u}_1$ act trivially on V and obtain the $(\mathfrak{g}, M)$ module $U(\mathfrak{g}) \otimes_{U(\mathfrak{q}_1)} V$. If π_τ is an irreducible representation of M with highest weight τ and representation space $F_M(\tau)$, we set

$$N_{\mathfrak{q}_1}(\tau) = U(\mathfrak{g}) \otimes_{U(\mathfrak{q}_1)} F_M(\tau). \tag{4.16}$$

(4.17) PROPOSITION. *Let $F_M(\tau)$ be the representation space of an irreducible representation of M. Then $N_{\mathfrak{q}_1}(\tau) \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} N_\mathcal{S}(\tau)$ as $(\mathfrak{g}, M)$ modules.*

PROOF. By (4.7) we know $N_\mathcal{S}(\tau) \cong U(\ell) \otimes_{U(\mathcal{P}_\ell)} F_M(\tau)$ as (ℓ, M) modules. By definition the action of $\mathfrak{u}$ on $N_\mathcal{S}(\tau)$ is trivial. Now by (4.1) one sees that $\mathfrak{u}_1 = \mathfrak{u} \oplus \mathfrak{u}_\mathcal{S}$ so that $\mathcal{P}_\ell \oplus \mathfrak{u} = \mathfrak{m} \oplus \mathfrak{u}_\mathcal{S} \oplus \mathfrak{u} = \mathfrak{m} \oplus \mathfrak{u}_1 = \mathfrak{q}_1$. We define $\mathfrak{u}$ to act trivially on $F_M(\tau)$ and consider the $(\mathfrak{q}, M)$ module $U(\mathfrak{q}) \otimes_{U(\mathcal{P}_\ell \oplus \mathfrak{u})} F_M(\tau) = U(\mathfrak{q}) \otimes_{U(\mathfrak{q}_1)} F_M(\tau)$. We observe that as (ℓ, M) modules, we have $U(\ell) \otimes_{U(\mathcal{P}_\ell)} F_M(\tau) \cong U(\mathfrak{q}) \otimes_{U(\mathfrak{q}_1)} F_M(\tau)$. Moreover, since $[\ell, \mathfrak{u}] \subset \mathfrak{u}$, it follows that the action of $\mathfrak{u}$ on $U(\mathfrak{q}) \otimes_{U(\mathfrak{q}_1)} F_M(\tau)$ is trivial. Since $\mathfrak{u}$ already acts trivially on $U(\ell) \otimes_{U(\mathcal{P}_\ell)} F_M(\tau)$ (by definition), it follows that $U(\ell) \otimes_{U(\mathcal{P}_\ell)} F_M(\tau) \cong U(\mathfrak{q}) \otimes_{U(\mathfrak{q}_1)} F_M(\tau)$ as $(\mathfrak{q}, M)$ modules. But then we have $(\mathfrak{g}, M)$ module isomorphisms

$$\begin{aligned} U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} N_\mathcal{S}(\tau) &\cong U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} (U(\ell) \otimes_{U(\mathcal{P}_\ell)} F_M(\tau)) \\ &\cong U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} (U(\mathfrak{q}) \otimes_{U(\mathfrak{q}_1)} F_M(\tau)) \\ &\cong U(\mathfrak{g}) \otimes_{U(\mathfrak{q}_1)} F_M(\tau) = N_{\mathfrak{q}_1}(\tau). \end{aligned}$$

(4.18) PROPOSITION. *Let the weight λ be analytically integral and $\Delta^+(\ell)$ dominant. Then there exists an exact sequence of $(\mathfrak{g}, M)$ modules*

$$0 \to N_{\mathfrak{q}_1}(\lambda_p) \to N_{\mathfrak{q}_1}(\lambda_{p-1}) \to \cdots \to N_{\mathfrak{q}_1}(\lambda) \to N_{\mathfrak{q}}(\lambda) \to 0.$$

PROOF. Let V be a $(\mathfrak{q}, M)$ module and set $F(V) = U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} V$. By (6.1.6) [p.302, 18], F is an exact covariant functor. Define $\mathfrak{u}$ to act trivially on the (ℓ, M) modules $C_j^{(\ell,M)}$, $0 \le j \le p$, in (4.13). By applying F to the exact sequence (4.14) we obtain an exact sequence of $(\mathfrak{g}, M)$ modules. The first $p+1$ nontrivial modules in this exact sequence are

$$\begin{aligned} F(C_j^{(\ell,M)}) &= U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} (U(\ell) \otimes_{U(\mathcal{P}_\ell)} F_M(\lambda_j)) \\ &\tilde{=} N_{\mathfrak{q}_1}(\lambda_j), \qquad 0 \le j \le p, \end{aligned}$$

by (4.17). The last nontrivial module is $F(F_L(\lambda)) = U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} F_L(\lambda) = N_{\mathfrak{q}}(\lambda)$ by (4.15). This completes the proof.

Let σ denote the conjugate linear antiautomorphism of $U(\mathfrak{g})$ induced by the map $x \to -x$ on the real form $\mathfrak{g}_0$ of $\mathfrak{g}$. Let E be an $(\mathfrak{a}, M)$ module. Let $\mathrm{Hom}_{\bar{\mathbb{C}}}(E, \mathbb{C})$ denote the space of conjugate linear maps $\varphi : E \to \mathbb{C}$. We define a $\sigma(\mathfrak{a}) = \bar{\mathfrak{a}}$ action and M action on $\mathrm{Hom}_{\bar{\mathbb{C}}}(E, \mathbb{C})$ by setting

$$\begin{aligned} (x \cdot \varphi)(e) &= \varphi(\sigma(x)e), \qquad x \in \bar{\mathfrak{a}}, e \in E, \\ (g \cdot \varphi)(e) &= \varphi(g^{-1} \cdot e), \qquad g \in M, e \in E. \end{aligned}$$

Let $\mathrm{Hom}_{\bar{\mathbb{C}}}(E, \mathbb{C})_M = \{\varphi \in \mathrm{Hom}_{\bar{\mathbb{C}}}(E, \mathbb{C}) : \dim(\mathrm{span}_{\mathbb{C}}\{g \cdot \varphi : g \in M\}) < \infty\}$ denote the space of M finite vectors. As in 6.1.20 [18], the functor $E \to \mathrm{Hom}_{\bar{\mathbb{C}}}(E, \mathbb{C})_M$ from $\mathcal{C}(\mathfrak{a}, M)$ to $\mathcal{C}(\bar{\mathfrak{a}}, M)$ is contravariant and exact.

Let W be a $(\bar{\mathfrak{q}}_1, M)$ module and let $\mathrm{Hom}_{U(\bar{\mathfrak{q}}_1)}(U(\mathfrak{g}), W)$ denote the space of $\mathbb{C}$ linear maps $\varphi : U(\mathfrak{g}) \to W$ that satisfy $\varphi(xy) = x \cdot \varphi(y)$ for all $x \in U(\bar{\mathfrak{q}}_1)$ and $y \in U(\mathfrak{g})$. We define an action of $\mathfrak{g}$ and M on $\mathrm{Hom}_{U(\bar{\mathfrak{q}}_1)}(U(\mathfrak{g}), W)$ by setting

$$\begin{aligned} (x \cdot \varphi)(y) &= \varphi(yx), \qquad x, y \in U(\mathfrak{g}), \\ (g \cdot \varphi)(y) &= g \cdot \varphi(\mathrm{Ad}(g^{-1})y), \quad g \in M, y \in U(\mathfrak{g}). \end{aligned}$$

We follow the notation of 6.1.21 [18] and let $\mathrm{pro}_{\bar{\mathfrak{q}}_1, M}^{\mathfrak{g}, M}(W) = \mathrm{Hom}_{U(\bar{\mathfrak{q}}_1)}(U(\mathfrak{g}), W)_M$ be the space of M finite vectors in the space $\mathrm{Hom}_{U(\bar{\mathfrak{q}}_1)}(U(\mathfrak{g}), W)$. Then by 6.1.22 [18], the functor $\mathrm{pro}_{\bar{\mathfrak{q}}_1, M}^{\mathfrak{g}, M} : \mathcal{C}(\bar{\mathfrak{q}}_1, M) \to \mathcal{C}(\mathfrak{g}, M)$ is covariant, exact and takes injectives to injectives.

Let F_M be a complex, finite-dimensional representation space for a representation π of M. Using the derived representation $d\pi$, the space F_M is an $\mathfrak{m}$ module. We may view F_M both as a $\mathfrak{q}_1$ module and a $\bar{\mathfrak{q}}_1$ module by defining the action of $\mathfrak{u}_1$ and $\bar{\mathfrak{u}}_1$ to be trivial. Since M is compact, there exists an inner product $< \cdot, \cdot >$ on F_M with $< g \cdot e, e' > = < e, g^{-1} \cdot e' >$ and $< x \cdot e, e' > = < e, \sigma(x)e' >$ for all $x \in U(\mathfrak{m})$, $g \in M$ and $e, e' \in F_M$. But since both $\mathfrak{u}_1$ and $\bar{\mathfrak{u}}_1$ act trivially on F_M, we have $< x \cdot e, e' > = < e, \sigma(x)e' >$ for all $x \in U(\bar{\mathfrak{q}}_1)$ as well.

(4.19) LEMMA. *Let F_M be a complex finite-dimensional representation space for a representation of M. View F_M as a $(\mathfrak{q}_1, M)$ module and as a $(\bar{\mathfrak{q}}_1, M)$ module by letting both $\mathfrak{u}_1$ and $\bar{\mathfrak{q}}_1$ act trivially. Then there exists a $(\bar{\mathfrak{q}}_1, M)$ module isomorphism $F_M \cong Hom_{\bar{\mathbb{C}}}(F_M, \mathbb{C})$.*

PROOF. Define the $\mathbb{C}$ linear map $F_M \to \mathrm{Hom}_{\bar{\mathbb{C}}}(F_M, \mathbb{C})$ by $e \mapsto \varphi_e$ where $\varphi(e') = < e, e' >$. This map is injective by the nondegeneracy of $< \cdot, \cdot >$ and surjective since $\dim(F_M) < \infty$. Moreover, by the invariance of $< \cdot, \cdot >$ this map is a $(\bar{\mathfrak{q}}_1, M)$ module isomorphism.

(4.20) PROPOSITION. *Let F_M be a complex finite-dimensional representation space for a representation of M. View F_M as a $(\mathfrak{q}_1, M)$ and as a $(\bar{\mathfrak{q}}_1, M)$ module by letting both $\mathfrak{u}_1$ and $\bar{u}_1$ act trivially. Then there exists a $(\mathfrak{g}, M)$ module isomorphism*

$$\mathrm{pro}^{\mathfrak{g}, M}_{\bar{\mathfrak{q}}_1, M}(F_M) \cong \mathrm{Hom}_{\bar{\mathbb{C}}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{q}_1)} F_M, \mathbb{C})_M.$$

PROOF. By a standard middle linear argument, there exists a $\mathbb{C}$ linear map $\Phi : \mathrm{Hom}_{U(\bar{\mathfrak{q}}_1)}(U(\mathfrak{g}), \mathrm{Hom}_{\bar{\mathbb{C}}}(F_M, \mathbb{C})) \to \mathrm{Hom}_{\bar{\mathbb{C}}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{q}_1)} F_M, \mathbb{C})$ such that

$$(\Phi\varphi)(x \otimes e) = \varphi(\sigma(x))(e),$$

for $x \otimes e \in U(\mathfrak{g}) \otimes F_M$ and $\varphi \in \mathrm{Hom}_{U(\bar{\mathfrak{q}}_1)}(U(\mathfrak{g}), \mathrm{Hom}_{\bar{\mathbb{C}}}(F_M, \mathbb{C}))$. We now show that Φ is a $(\mathfrak{g}, M)$ map. Let $\varphi \in \mathrm{Hom}_{U(\bar{\mathfrak{q}}_1)}(U(\mathfrak{g}), \mathrm{Hom}_{\bar{\mathbb{C}}}(F_M, \mathbb{C}))$, $x, y \in U(\mathfrak{g})$ and $e \in F_M$. Then $\Phi(x \cdot \varphi)(y \otimes e) = (x \cdot \varphi)(\sigma(y))(e) = \varphi(\sigma(y)x)(e) = \varphi(\sigma(\sigma(x)y))(e) = (\Phi(\varphi))(\sigma(x)y \otimes e) = (x \cdot \Phi(\varphi))(y \otimes e)$. Thus, $\Phi(x \cdot \varphi) = x \cdot \Phi(\varphi)$ for all $x \in U(\mathfrak{g})$. Let $g \in M$. Then we have $(\Phi(g \cdot \varphi))(y \otimes e) = (g \cdot \varphi)(\sigma(y))(e) = (g \cdot \varphi(\mathrm{Ad}(g^{-1})\sigma(y)))(e) = \varphi(\sigma(\mathrm{Ad}(g^{-1})y))(g^{-1} \cdot e) = (\Phi\varphi)(\mathrm{Ad}(g^{-1})y \otimes g^{-1} \cdot e) = (g \cdot \Phi\varphi)(y \otimes e)$. Thus, $\Phi(g \cdot \varphi) = g \cdot \Phi(\varphi)$ for all $g \in M$ so that Φ is a $(\mathfrak{g}, M)$ map.

Define $\Psi : \mathrm{Hom}_{\bar{\mathbb{C}}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{q}_1)} F_M, \mathbb{C}) \to \mathrm{Hom}_{U(\bar{\mathfrak{q}}_1)}(U(\mathfrak{g}), \mathrm{Hom}_{\bar{\mathbb{C}}}(F_M, \mathbb{C}))$ by

$$(\Psi(\chi)(x))(e) = \chi(\sigma(x) \otimes e)$$

for $\chi \in \mathrm{Hom}_{\bar{\mathbb{C}}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{q}_1)} F_M, \mathbb{C})$, $x \in U(\mathfrak{g})$ and $e \in F_M$. Observe that the map $e \mapsto (\Psi(\chi)(x))(e)$ is conjugate linear. Moreover, if $q \in U(\bar{\mathfrak{q}}_1)$, then $\Psi(\chi)(qx)(e) = \chi(\sigma(qx) \otimes e) = \chi(\sigma(x) \otimes \sigma(q)e) = (\Psi(\chi)(x))(\sigma(q)e) = (q \cdot \Psi(\chi)(x))(e)$. Thus $\Psi(\chi)(qx) = q \cdot \Psi(\chi)(x)$ so we conclude that $\Psi(\chi) \in \mathrm{Hom}_{U(\bar{\mathfrak{q}}_1)}(U(\mathfrak{g}), \mathrm{Hom}_{\bar{\mathbb{C}}}(F_M, \mathbb{C}))$. It is readily checked that Ψ is the inverse of Φ. Since M finite vectors are clearly mapped to M finite vectors, there exists a $(\mathfrak{g}, M)$ module isomorphism

$$\mathrm{Hom}_{U(\bar{\mathfrak{q}}_1)}(U(\mathfrak{g}), \mathrm{Hom}_{\bar{\mathbb{C}}}(F_M, \mathbb{C}))_M \cong \mathrm{Hom}_{\bar{\mathbb{C}}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{q}_1)} F_M, \mathbb{C})_M.$$

By (4.19), we have $\mathrm{Hom}_{\bar{\mathbb{C}}}(F_M, \mathbb{C}) \cong F_M$ so that the proposition now follows.

(4.21) Remarks. (1) If F_L is a complex, finite-dimensional representation of L, then by restricting the L action to M and letting $\mathfrak{u}$ and $\bar{\mathfrak{u}}$ act trivially on F_L, we may argue as above to conclude

$$\mathrm{pro}^{\mathfrak{g}, M}_{\bar{\mathfrak{q}}, M}(F_L) \cong \mathrm{Hom}_{\bar{\mathbb{C}}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} F_L, \mathbb{C})_M.$$

(2) Let W and V be two $(\mathfrak{g}, M)$ modules. We say W is conjugate dual to V if there exists a nondegenerate sesquilinear pairing $< \cdot, \cdot >$ of V and W such that $< g \cdot v, w >=< v, g^{-1} \cdot w >$ and $< x \cdot v, w >=< v, \sigma(x)w >$ for all $v \in V$, $w \in W$, $x \in U(\mathfrak{g})$ and $g \in M$. The proof of (4.20) then shows that $\operatorname{pro}_{\mathfrak{q}_1,M}^{\mathfrak{g},M}(F_M)$ is conjugate dual to $U(\mathfrak{g}) \otimes_{U(\mathfrak{q}_1)} F_M$. The appropriate pairing in this case is $< \varphi, x \otimes e >=< \varphi(\sigma(x)), e >$, where $< \cdot, \cdot >$ on the right side denotes the invariant inner product on F_M.

(4.22) PROPOSITION. *Let the weight λ be analytically integral and $\Delta^+(\ell)$ dominant. There exists an exact sequence of $(\mathfrak{g}, M)$ modules*

$$0 \to \operatorname{pro}_{\mathfrak{q},M}^{\mathfrak{g},M}(F_L(\lambda)) \to \operatorname{pro}_{\mathfrak{q}_1,M}^{\mathfrak{g},M}(F_M(\lambda)) \to \cdots \to \operatorname{pro}_{\mathfrak{q}_1,M}^{\mathfrak{g},M}(F_M(\lambda_p)) \to 0.$$

PROOF. Apply the contravariant exact functor $\operatorname{Hom}_{\mathbb{C}}(\cdot, \mathbb{C})_M$ to the exact sequence in (4.18) to obtain an exact sequence whose last $p+1$ nontrivial modules are, by (4.20),

$$\operatorname{Hom}_{\mathbb{C}}(N_{\mathfrak{q}_1}(\lambda_j), \mathbb{C})_M \tilde{=} \operatorname{pro}_{\mathfrak{q}_1,M}^{\mathfrak{g},M}(F_M(\lambda_j)), \qquad 0 \le j \le p,$$

and whose first nontrivial module is $\operatorname{pro}_{\mathfrak{q},M}^{\mathfrak{g},M}(F_L(\lambda))$ by (4.21.1).

The resolution in (4.21) is called the BGG resolution of $\operatorname{pro}_{\mathfrak{q},M}^{\mathfrak{g},M}(F_L(\lambda))$. We now state (3.2) in the case where the module A is the $(\mathfrak{g}, M)$ module $\operatorname{pro}_{\mathfrak{q},M}^{\mathfrak{g},M}(F_L(\lambda))$, the resolution of A is the resolution given in (4.22) and the functor $\Gamma = \Gamma_{\mathfrak{g},M}^{\mathfrak{g},K}$ is the Zuckerman functor. We refer the reader to section 6.2 of [18] for the precise definition of Γ and some relevant properties. In particular, by (6.2.10) [18] Γ is left exact and the category $\mathcal{C}(\mathfrak{g}, M)$ has enough injectives so that the right derived functors $\Gamma^i, i \ge 0$, are well defined (cf. (6.2.11) [18]).

(4.23) THEOREM. *Let the weight λ be analytically integral and $\Delta^+(\ell)$ dominant and let $F_L(\lambda)$ denote the representation space for the irreducible representation of L with highest weight λ. Let N denote the deleted complex*

$$0 \to \operatorname{pro}_{\mathfrak{q}_1,M}^{\mathfrak{g},M}(F_M(\lambda)) \to \cdots \to \operatorname{pro}_{\mathfrak{q}_1,M}^{\mathfrak{g},M}(F_M(\lambda_p)) \to 0.$$

Then there exists a first quadrant spectral sequence

$$E_2^{n,m} = H^n(\Gamma^m(N)) \underset{n}{\Longrightarrow} \Gamma^{n+m}\left(\operatorname{pro}_{\mathfrak{q},M}^{\mathfrak{g},M}(F_L(\lambda))\right).$$

As a first step in determining the E_2 terms in (4.23), we now compute $\Gamma^m(N)$. By this we mean that we now identify each of the $(\mathfrak{g}, K)$ modules $\Gamma^m(\operatorname{pro}_{\mathfrak{q}_1,M}^{\mathfrak{g},M}(F_M(\lambda_j)))$ for all $0 \le j, m$ where $\lambda_0 = \lambda$. Recall the parabolic $\mathfrak{q}(\omega_p) = \mathfrak{k} \oplus \mathfrak{p}_+$ of $\mathfrak{g}$. We put $\mathcal{P}_k = \mathfrak{q}(\omega_{p+1}) \cap \mathfrak{k}$. Then $\mathcal{P}_k$ is a parabolic subalgebra of $\mathfrak{k}$.

(4.24) PROPOSITION. *If V is a $(\mathfrak{k}, M)$ module and W is an $(\mathfrak{m}, M)$ module, then one has*

$$\operatorname{pro}_{\mathfrak{q}(\omega_p),K}^{\mathfrak{g},K}\left(\Gamma_{\mathfrak{k},M}^{\mathfrak{k},K}(V)\right) = \Gamma_{\mathfrak{g},M}^{\mathfrak{g},K}\left(\operatorname{pro}_{\mathfrak{q}(\omega_p),M}^{\mathfrak{g},M}(V)\right), \tag{1}$$

$$\text{(2)} \qquad \operatorname{pro}^{\mathfrak{g},M}_{\overline{\mathfrak{q}(\omega_p)},M}\left(\operatorname{pro}^{\mathfrak{k},M}_{\overline{\mathcal{P}_k},M}(W)\right) = \operatorname{pro}^{\mathfrak{g},M}_{\bar{\mathfrak{q}}_1,M}(W).$$

PROOF. The first equation is a special case of (a) in (6.3.9) [18] where we have replaced $\mathfrak{q}^1$ and $\mathfrak{q}^2$ (in the notation of (6.3.6)) with $\bar{\mathfrak{q}}_1$ and $\overline{\mathfrak{q}(\omega_p)}$, respectively. This results in $L^1 \cap K = M, L^2 \cap K = K$ and $\mathfrak{q}^0 = \overline{\mathcal{P}_k}$. The second equation follows from (6.3.8) [18] with these same opposite parabolics. The verification of this, which left to the reader, is somewhat simpler since no tensoring with D, D^1 or D^2 is involved.

Let $\mathcal{W}_k$ denote the subgroup of $\mathcal{W}$ generated by the simple reflections $\{\sigma_i : i \neq p\}$. Then $\mathcal{W}_k$ is the Weyl group for K. Let ρ_k denote half the sum of positive roots $\Delta^+(\mathfrak{k})$ of $\mathfrak{k}$. A weight τ is said to be $\mathfrak{k}$ singular if $\tau + \rho_k$ is orthogonal to some root $\alpha \in \Delta(\mathfrak{k})$. If τ is $\mathfrak{k}$ nonsingular it follows that there exists $\sigma \in \mathcal{W}_k$ such that $\sigma(\tau + \rho_k)$ lies in the positive Weyl chamber, i.e., $(\sigma(\tau + \rho_k), \alpha) > 0$ for all $\alpha \in \Delta^+(\mathfrak{k})$. Moreover, the element $\sigma \in \mathcal{W}_k$ satisfying this property is unique.

Let $\tau \in \mathfrak{t}^*$ be analytically integral and $\Delta^+(\mathfrak{k})$ dominant. Since K is a compact linear connected reductive group, there exists a representation of K on a space $F_K(\tau)$ with highest weight τ. Suppose τ is analytically integral and $\mathfrak{k}$ nonsingular. By replacing $\mathcal{W}_\ell$ and ℓ by $\mathcal{W}_k$ and $\mathfrak{k}$ in (4.8), it follows that $\sigma(\tau+\rho_k)-\rho_k$ is analytically integral for all $\sigma \in \mathcal{W}_k$. If $\sigma \in \mathcal{W}_k$ is chosen to be the unique element for which $\sigma(\tau+\rho_k)-\rho_k$ is $\Delta^+(\mathfrak{k})$ dominant, then the K type $F_K(\sigma(\tau+\rho_k)-\rho_k)$ is well defined.

We now state the Bott-Borel-Weil theorem for the compact group K in a manner convenient for our purposes. The proof appears in [10]. The version given here is the conjugate dual version of the theorem given in (6.2) [9]. One applies (6.3) [9] to (6.2) [9] to obtain the version below. The version of the theorem for the group $U(n)$ in (7.35) [12] is stated in terms of opposite parabolics and is comparable in form to the version given here.

(4.25) PROPOSITION. *Suppose τ is analytically integral and $\Delta^+(\mathfrak{m})$ dominant. Let $F_M(\tau)$ denote the representation space of the irreducible representation of M with highest weight τ.*

(1) *If τ is $\mathfrak{k}$ singular, then $\left(\Gamma^{\mathfrak{k},K}_{\mathfrak{k},M}\right)^j(\operatorname{pro}^{\mathfrak{k},M}_{\overline{\mathcal{P}_k},M}(F_M(\tau)))$ is zero for all $j \geq 0$.*

(2) *Let τ be $\mathfrak{k}$ nonsingular and let $\sigma \in \mathcal{W}_k$ denote the unique element for which $\sigma(\tau+\rho_k)-\rho_k$ is $\Delta^+(\mathfrak{k})$ dominant. Then*

$$\left(\Gamma^{\mathfrak{k},K}_{\mathfrak{k},M}\right)^j(\operatorname{pro}^{\mathfrak{k},M}_{\overline{\mathcal{P}_k},M}(F_M(\tau))) = \begin{cases} F_K(\sigma(\tau+\rho_k)-\rho_k) & \text{if } j = \ell(\sigma), \\ 0 & \text{otherwise .} \end{cases}$$

(4.26) PROPOSITION. *Suppose τ is analytically integral and $\Delta^+(\mathfrak{m})$ dominant. Let $F_M(\tau)$ denote the representation space of the irreducible representation of M with highest weight τ.*

(1) *If τ is $\mathfrak{k}$ singular, then $\Gamma^j(\operatorname{pro}^{\mathfrak{g},M}_{\bar{\mathfrak{q}}_1,M}(F_M(\tau)))$ is zero for all $j \geq 0$.*

(2) *Let τ be $\mathfrak{k}$ nonsingular and let $\sigma \in \mathcal{W}_k$ denote the unique element for which $\sigma(\tau+\rho_k)-\rho_k$ is $\Delta^+(\mathfrak{k})$ dominant. Then*

$$\Gamma^j(\mathrm{pro}^{\mathfrak{g},M}_{\mathfrak{q}_1,M}(F_M(\tau))) = \begin{cases} \mathrm{pro}^{\mathfrak{g},K}_{\mathfrak{q}(\omega_p),K}(F_K(\sigma(\tau+\rho_k)-\rho_k)) & \text{if } j=\ell(\sigma), \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Recall that the functor pro is covariant, exact and carries injectives to injectives. We then have

$$\begin{aligned} \Gamma^j(\mathrm{pro}^{\mathfrak{g},M}_{\mathfrak{q}_1,M}(F_M(\tau))) &= \Gamma^j\Big(\mathrm{pro}^{\mathfrak{g},M}_{\mathfrak{q}(\omega_p),M}\Big(\mathrm{pro}^{\mathfrak{k},M}_{\mathcal{P}_k,M}(F_M(\tau))\Big)\Big) && \text{by (4.24.2)} \\ &= \Big(\Gamma\circ\mathrm{pro}^{\mathfrak{g},M}_{\mathfrak{q}(\omega_p),M}\Big)^j\Big(\mathrm{pro}^{\mathfrak{k},M}_{\mathcal{P}_k,M}(F_M(\tau))\Big) && \text{since pro is exact} \\ &= \Big(\mathrm{pro}^{\mathfrak{g},K}_{\mathfrak{q}(\omega_p),K}\circ\Gamma^{\mathfrak{k},K}_{\mathfrak{k},K}\Big)^j\Big(\mathrm{pro}^{\mathfrak{k},M}_{\mathcal{P}_k,M}(F_M(\tau))\Big) && \text{by (4.24.1)} \\ &= \mathrm{pro}^{\mathfrak{g},K}_{\mathfrak{q}(\omega_p),K}\circ\big(\Gamma^{\mathfrak{k},K}_{\mathfrak{k},M}\big)^j\Big(\mathrm{pro}^{\mathfrak{k},M}_{\mathcal{P}_k,M}(F_M(\tau))\Big) && \text{since pro is exact.} \end{aligned}$$

If τ is $\mathfrak{k}$ singular, then this last line is zero by (4.25.1). This proves (1). If τ is $\mathfrak{k}$ nonsingular and $\sigma \in \mathcal{W}_k$ is chosen as in (2), then by (4.25.2) this last line is $\mathrm{pro}^{\mathfrak{g},K}_{\mathfrak{q}(\omega_p),K}(F_K(\sigma(\tau+\rho_k)-\rho_k))$ if $j=\ell(\sigma)$ and 0 otherwise. This proves (2).

5. Computation of E_2 terms

For certain highest weights λ, the E_2 terms of the spectral sequence in (4.23) will now be determined. The weights λ_j in (4.9) are first written in a form which allows us to easily identify which of these are $\mathfrak{k}$ nonsingular. For the $\mathfrak{k}$ nonsingular λ_j, the explicit form of the $\Delta^+(\mathfrak{k})$ dominant weights $\sigma(\lambda_j+\rho_k)-\rho_k$ is then obtained. By plotting these $\Delta^+(\mathfrak{k})$ dominant weights we obtain a table whose m^{th} row may be identified with the complex $\Gamma^m(N)$ where N is given in (4.23). Since the complex $\Gamma^m(N)$ is known explicitly by (4.26), the desired E_2 terms result by taking cohomology of the rows. In addition, certain higher order terms of this spectral sequence play a crucial role in what we do in section six so we proceed to determine these terms as well. We begin with some routine calculations.

Recall that σ_i is Weyl group element corresponding to the simple root α_i, $1 \le i \le n-1$. We adopt the convention that if an index j occurs for which ω_j is undefined, then that term is omitted from the expression.

(5.1) LEMMA. *Let $\lambda = \sum_{i=1}^{n-1} a_i\omega_i$, where $a_i \in \mathbb{R}$. Let $s \ge 1$ and $t \ge 0$ be integers satisfying $s+t \le n-1$ and set $\sigma = \sigma_{s+t}\cdots\sigma_s$. We have*

(1)

$$\sigma(\lambda) = \sum_{i=1}^{s-1} a_i\omega_i + \sum_{i=s-1}^{s+t-1} a_{i+1}\omega_i - \Big(\sum_{i=s}^{s+t} a_i\Big)\omega_{s+t} + \Big(\sum_{i=s}^{s+t} a_i\Big)\omega_{s+t+1} + \sum_{i=s+t+1}^{n-1} a_i\omega_i,$$

(2)

$$\sigma(\lambda)-\lambda = a_s\omega_{s-1} + n\sum_{j=s}^{s+t-1}(a_{j+1}-a_j)\omega_j - \Big(a_{s+t}+\sum_{i=s}^{s+t} a_i\Big)\omega_{s+t} + \Big(\sum_{i=s}^{s+t} a_i\Big)\omega_{s+t+1}.$$

PROOF. Using the fact $\sigma_i(\omega_j) = \omega_j - \delta_{ij}\alpha_i$, it readily follows by induction on t that $\sigma(\lambda)$ has the form given in (1). The expression for $\sigma(\lambda) - \lambda$ in (2) then follows.

(5.2) DEFINITION. Let $\lambda = \sum_{i=1}^{n-1} a_i\omega_i$ and for each integer j satisfying $1 \leq j \leq p$ set $\lambda_j = \sigma_p \cdots \sigma_{p-j+1}(\lambda + \rho) - \rho$. We also put $\lambda_0 = \lambda$.

Since $\rho - \rho_\ell$ is a multiple of ω_{p+1} and since each product $\sigma_p \cdots \sigma_{p-j+1}$ fixes ω_{p+1}, it follows that $\lambda_j = \sigma_p \cdots \sigma_{p-j+1}(\lambda + \rho_\ell) - \rho_\ell$. Thus, the form of the weights λ_j defined in (5.2) agree with the weights defined in (4.9).

(5.3) LEMMA. *For each integer j satisfying $1 \leq j \leq p$ the weight λ_j has the form*

$$\lambda_j = \omega_{p-j} + \sum_{i=1}^{p-j} a_i\omega_i + \sum_{i=p-j}^{p-1} a_{i+1}\omega_i - \Big(\sum_{i=p-j+1}^{p} a_i + j + 1\Big)\omega_p + \Big(\sum_{i=p-j+1}^{p+1} a_i + j\Big)\omega_{p+1} + \sum_{i=p+2}^{n-1} a_i\omega_i.$$

PROOF. With $s = p - j + 1$ and $t = j - 1$, it follows by (5.1.2) that $\sigma_p \cdots \sigma_{p-j+1}(\rho) - \rho = \omega_{p-j} - (j+1)\omega_p + j\omega_{p+1}$ and by (5.1.1) that

$$\sigma_p \cdots \sigma_{p-j+1}(\lambda) = \sum_{i=1}^{p-j} a_i\omega_i + \sum_{i=p-j}^{p-1} a_{i+1}\omega_i - \Big(\sum_{i=p-j+1}^{p} a_i\Big)\omega_p + \Big(\sum_{i=p-j+1}^{p} a_i\Big)\omega_{p+1} + \sum_{i=p+1}^{n-1} a_i\omega_i.$$

The lemma follows by adding these and combining coefficients.

(5.4) DEFINITION. If $\lambda = \sum_{i=1}^{n-1} a_i\omega_i$ and λ_j, $0 \leq j \leq p$, is defined as in (5.2), then for each $j \in \{0, \ldots, p\}$ and $\nu \in \{1, \ldots, q-1\}$ define $A_\nu^j = A_\nu^j(\lambda)$ by $A_\nu^j = -(\lambda_j + \rho_k, \alpha_{p+1} + \cdots + \alpha_{p+\nu})$. By (5.3) we have

$$A_\nu^j = -\nu - j - \sum_{i=p-j+1}^{p+\nu} a_i.$$

Motivated by (4.26), we now determine the $\mathfrak{k}$ singularity of the weights λ_j that appear in the complex N in (4.23). Recall that the weight $\lambda = \sum_{i=1}^{n-1} a_i\omega_i$ is integral if and only if $a_i \in \mathbb{Z}$ for $i \in \{1, \ldots, n-1\}$ and is dominant for $\Delta^+(\ell)$ if and only if $a_i \geq 0$ for $i \neq p+1$.

(5.5) PROPOSITION. *Assume $\lambda = \sum_{i=1}^{n-1} a_i\omega_i$ is integral and $\Delta^+(\ell)$ dominant.*

(1) *The functions $j \mapsto A_\nu^j$ and $\nu \mapsto A_\nu^j$ are strictly decreasing $\mathbb{Z}$ valued functions.*

(2) *The weight λ_j is $\mathfrak{k}$ singular if and only if there exists a $\nu \in \{1, \ldots, q-1\}$ such that $A_\nu^j = 0$. The value of ν with this property is unique.*

(3) *The weight* λ_j *is* $\Delta^+(\mathfrak{k})$ *dominant if and only if* $A_1^j < 0$.

(4) *If* λ_j *is* $\mathfrak{k}$ *nonsingular and not* $\Delta^+(\mathfrak{k})$ *dominant, then there exists* $\nu \in \{1, \dots, q-1\}$ *such that* $A_\nu^j > 0$. *Define* ν_j *to be the greatest such integer. Then* $\sigma_{p+\nu_j} \cdots \sigma_{p+1}(\lambda_j + \rho_k) - \rho_k$ *is* $\Delta^+(\mathfrak{k})$ *dominant.*

PROOF. Part (1) is clear by the explicit form of A_ν^j given in (5.4) and the assumption that λ is integral and $\Delta^+(\ell)$ dominant. Since λ is $\Delta^+(\ell)$ dominant, a_{p+1} is the only possible negative coefficient in λ. By the form of λ_j given in (5.3), it is clear that $(\lambda_j + \rho_k, \alpha) = 0$ for some $\alpha \in \Delta^+(\mathfrak{k})$ if and only if $\alpha = \alpha_{p+1} + \cdots + \alpha_{p+\nu}$ for some $\nu \in \{1, \dots, q-1\}$. If such a ν exists, it must be unique by (1). This proves (2).

To prove (3), note by (5.3) that λ_j is $\Delta^+(\mathfrak{k})$ dominant if and only if $(\lambda_j, \alpha_{p+1}) \geq 0$. But $A_1^j = -(\lambda_j + \rho_k, \alpha_{p+1})$ so that $A_1^j < 0$ if and only if $(\lambda_j, \alpha_{p+1}) > -(\rho_k, \alpha_{p+1}) = -1$. Since λ is integral, so is λ_j by (5.3). Consequently, $(\lambda_j, \alpha_{p+1})$ is an integer. This proves (3).

Now suppose λ_j is $\mathfrak{k}$ nonsingular and not $\Delta^+(\mathfrak{k})$ dominant. By (2) we know that $A_\nu^j \neq 0$ for all $\nu \in \{1, \dots, q-1\}$. By (3), we must have $A_1^j > 0$ so that ν_j in (4) is well defined. Then ν_j satisfies $A_{\nu_j}^j > 0$ and, if $\nu_j < q-1$, $A_{\nu_j+1}^j < 0$ by (1). Since each σ_i with $i \neq p$ fixes ω_p, we have

$$\begin{aligned}\sigma_{p+\nu_j} \cdots \sigma_{p+1}(\lambda_j + \rho_k) - \rho_k &= \sigma_{p+\nu_j} \cdots \sigma_{p+1}(\lambda_j + \rho) - \rho \\ &= \sigma_{p+\nu_j} \cdots \sigma_{p+1}(\sigma_p \cdots \sigma_{p-j+1}(\lambda + \rho)) - \rho\end{aligned}$$

by the definition of λ_j. By (5.1.1) we have

$$\begin{aligned}\sigma_{p+\nu_j} \cdots \sigma_{p-j+1}(\lambda) = \sum_{i=1}^{p-j} a_i\omega_i + \sum_{i=p-j}^{p+\nu_j-1} a_{i+1}\omega_i - \Big(\sum_{i=p-j+1}^{p+\nu_j} a_i\Big)\omega_{p+\nu_j} \\ + \Big(\sum_{i=p-j+1}^{p+\nu_j+1} a_i\Big)\omega_{p+\nu_j+1} + \sum_{i=p+\nu_j+2}^{n-1} a_i\omega_i\end{aligned}$$

and by (5.1.2) we have $\sigma_{p+\nu_j} \cdots \sigma_{p-j+1}(\rho) - \rho = \omega_{p-j} - (\nu_j + j + 1)\omega_{p+\nu_j} + (\nu_j + j)\omega_{p+\nu_j+1}$. If $\nu_j < q-1$, then by (5.4) one has

$$\begin{aligned}\sigma_{p+\nu_j} \cdots \sigma_{p+1}(\lambda_j + \rho_k) - \rho_k = \omega_{p-j} + \sum_{i=1}^{p-j} a_i\omega_i + \sum_{i=p-j}^{p+\nu_j-1} a_{i+1}\omega_i \\ + (A_{\nu_j}^j - 1)\omega_{p+\nu_j} - (A_{\nu_j+1}^j + 1)\omega_{p+\nu_j+1} + \sum_{i=p+\nu_j+2}^{n-1} a_i\omega_i.\end{aligned}$$

Since both $(A_{\nu_j}^j - 1)$ and $-(A_{\nu_j+1}^j + 1)$ are nonnegative integers, it follows that $\sigma_{p+\nu_j} \cdots \sigma_{p+1}(\lambda_j + \rho_k) - \rho_k$ is $\Delta^+(\mathfrak{k})$ dominant. The $\nu_j = q-1$ case is similar and its proof is omitted. This completes the proof of (4).

(5.6) COROLLARY. *Assume λ is integral and $\Delta^+(\ell)$ dominant.*

(1) *If $A^j_{q-1} > 0$ for some $j \in \{0, \dots, p\}$, then each λ_r with $r \leq j$ is $\mathfrak{k}$ nonsingular and not $\Delta^+(\mathfrak{k})$ dominant. Moreover, the integer ν_r defined in (5.5.4) is $q-1$.*

(2) *If there exists $\nu' \in \{1, \dots, q-2\}$ satisfying $A^j_{\nu'} > 0$ and $A^j_{\nu'+1} < 0$, then λ_j is $\mathfrak{k}$ nonsingular and not $\Delta^+(\mathfrak{k})$ dominant. Moreover, ν_j in (5.5.4) is ν'.*

PROOF. Suppose $A^j_{q-1} > 0$ for some $j \in \{0, \dots, p\}$. From (5.5.1), we know $A^r_{q-1} \geq A^j_{q-1}$ for all $r \leq j$. Thus, $A^r_{q-1} > 0$ so by (5.5.2) we conclude λ_r is $\mathfrak{k}$ nonsingular. We also know $\lambda_r, r \leq j$, is not $\Delta^+(\mathfrak{k})$ dominant by (5.5.3). Finally, by the definition of ν_r it is clear that $\nu_r = q-1$. This proves (1).

Suppose $A^j_{\nu'} > 0$ and $A^j_{\nu'+1} < 0$ for some $\nu' \in \{1, \dots, q-2\}$. By (5.5.1) it follows that A^j_ν is never zero. Hence λ_j is $\mathfrak{k}$ nonsingular by (5.5.2) and is not $\Delta^+(\mathfrak{k})$ dominant by (5.5.3). Finally, by the definition of ν_j, it is clear that $\nu_j = \nu'$. This establishes (2) and so completes the proof.

We now introduce notation used to label the reduction points in the Enright, Howe and Wallach classification of unitary highest weight representations of $SU(p+1, q-1)$ [8]. We are interested in the reductions points since these weights are precisely the highest weights in the classsification for which the produced module $\mathrm{pro}^{\mathfrak{g},M}_{\mathfrak{q},M}(F_L(\lambda))$ resolved in (4.22) is reducible. The reducibility of $\mathrm{pro}^{\mathfrak{g},M}_{\mathfrak{q},M}(F_L(\lambda))$ is essential here for otherwise (by Theorem (6.3.5) and Proposition (6.4.6) [19]) the modules $\Gamma^j(\mathrm{pro}^{\mathfrak{g},M}_{\mathfrak{q},M}(F_L(\lambda))$ are zero for $j \neq S$, the middle dimension. Since we wish to study the derived functors Γ^j for $j < S$, we naturally begin with reducible modules.

By tables (6.21) and (6.22) [5], the reduction points for $SU(p+1, q-1)$ are $\Delta^+(\ell)$ dominant integral weights of the form

$$\lambda = \sum_{i=p'}^{p} a_i\omega_i - (n - p' - q' + \ell - 1 + \sum_{i \neq p+1} a_i)\omega_{p+1} + \sum_{i=p+2}^{n-q'} a_i. \tag{5.7}$$

Here the integer p' is defined as follows: if there exists an index i in $\{1, \dots, p\}$ such that $a_i \neq 0$, then p' is defined to be the smallest such integer. If $a_i = 0$ for all $i \in \{1, \dots, p\}$ we set $p' = p+1$. Similarly, if there exists an index $i \in \{1, \dots, q-2\}$ such that $a_{n-i} \neq 0$, then q' is defined to be the smallest such integer. In the case where $a_{n-i} = 0$ for all $i \in \{1, \dots, q-2\}$ we set $q' = q-1$. The integer $\ell > 0$ is called the level of reduction of λ and must satisfy $1 \leq \ell \leq \min\{p', q'\}$ in order for the $\Delta^+(\ell)$ dominant integral weight in (5.7) to be a reduction point. Moreover, since each reduction point is the highest weight of an irreducible representation of L, it is therefore analytically integral. Throughout the rest of this article we work only with λ in (5.7) that are reduction points.

(5.8) PROPOSITION. *Assume λ in (5.7) is a reduction point.*

(1) *If λ_j is $\mathfrak{k}$ nonsingular and $\nu_j < q-1$, then $j \geq p-p'+\ell$ and $\nu_j \leq q-q'-1$.*

(2) *The weight $\lambda_{p-p'+\ell}$ is $\mathfrak{k}$ nonsingular. If $q-q'-1=0$, then $\lambda_{p-p'+\ell}$ is $\Delta^+(\mathfrak{k})$ dominant integral. If $q-q'-1>0$, then $\sigma_{n-q'-1}\cdots\sigma_{p+1}(\lambda_{p-p'+\ell}+\rho_k)-\rho_k$ is $\Delta^+(\mathfrak{k})$ dominant integral.*

PROOF. Suppose λ_j is $\mathfrak{k}$ nonsingular and $\nu_j < q-1$. By (5.6.2), we know $A^j_{\nu_j+1}+1 < A^j_{\nu_j}$, i.e., $1 < A^j_{\nu_j} - A^j_{\nu_j+1} = 1 + a_{p+1+\nu_j}$. Since $0 < a_{p+\nu_j+1}$ and $a_{p+r}=0$ for $r > q-q'$ we conclude $\nu_j \leq q-q'-1$. Now consider the constraint $A^j_{\nu_j+1}+1 \leq 0$ in (5.6.2). This inequality is equivalent to

$$(p-p'+\ell-j)+(q-q'-1-\nu_j)+\sum_{i=1}^{p-j} a_i + \sum_{i=p+\nu_j+2}^{n-1} a_i \leq 0.$$

By assumption, each a_i appearing in this sum is nonnegative. Since we also have $\nu_j \leq q-q'-1$, we conclude $p-p'+\ell \leq j$. This proves (1).

To prove (2), we first consider the case $q-q'-1=0$. Then

$$A_1^{p-p'+\ell} = -1 + \sum_{i=1}^{p'-\ell} a_i + \sum_{i=n-q'+1}^{n-1} a_i = -1 < 0$$

so that by (5.5.3) we know $\lambda_{p-p'+\ell}$ is $\Delta^+(\mathfrak{k})$ dominant. Now suppose that $q-q'-1>0$. Then

$$A^{p-p'+\ell}_{q-q'-1} = \sum_{i=1}^{p'-\ell} a_i + \sum_{i=n-q'}^{n-1} a_i = a_{n-q'} > 0$$

and

$$A^{p-p'+\ell}_{q-q'} = -1 + \sum_{i=n-q'+1}^{n-1} a_i = -1.$$

Thus $\nu_{p-p'+\ell} = q-q'-1$ by (5.6.2) so $\sigma_{n-q'-1}\cdots\sigma_{p+1}(\lambda_{p-p'+\ell}+\rho_k)-\rho_k$ is $\Delta^+(\mathfrak{k})$ dominant by (5.5.4). This completes the proof of (2).

We continue to determine all the values of j between $p-p'+\ell$ and p for which λ_j is $\mathfrak{k}$ singular. We therefore assume $\ell < p'$. If $q-q'-1>0$, define the set $\underline{S}$ to be

$$\underline{S} = \left\{ s_t = t + \sum_{i=n-q'-t}^{n-q'} a_i : 0 \leq t \leq q-q'-2 \right\}.$$

Note that $\underline{S}$ is a strictly increasing sequence of $q-q'-1$ positive integers.

(5.9) PROPOSITION. *Let λ in (5.7) be a reduction point and assume $\ell < p'$.*

(1) *If $q-q'-1=0$, then $\lambda_{p-p'+\ell+s}$ is $\Delta^+(\mathfrak{k})$ dominant integral for each $s \in \{1,\ldots,p'-\ell\}$.*

(2) *If $q-q'-1>0$ and $s \in \{1,\ldots,p'-\ell\}$ lies in $\underline{S}$, then $\lambda_{p-p'+\ell+s}$ is $\mathfrak{k}$ singular.*

(3) *Let* $q-q'-1>0$ *and* $s_t, s_{t+1} \in \underline{S}$. *If* $s \in \{1,\dots,p'-\ell\}$ *lies between* s_t *and* s_{t+1}, *then the weight* $\lambda_{p-p'+\ell+s}$ *is* $\mathfrak{k}$ *nonsingular and* $\sigma_{n-q'-t-2}\cdots\sigma_{p+1}(\lambda_{p-p'+\ell+s}+\rho_k)-\rho_k$ *is* $\Delta^+(\mathfrak{k})$ *dominant integral. If* $s \in \{1,\dots,p'-\ell\}$ *satisfies* $s > s_{q-q'-2} = \max(\underline{S})$, *then* $\lambda_{p-p'+\ell+s}$ *is* $\Delta^+(\mathfrak{k})$ *dominant integral.*

(4) *If* $q-q'-1>0$ *and* $s \in \{1,\dots,p'-\ell\}$, *then* $\lambda_{p-p'+\ell+s}$ *is* $\mathfrak{k}$ *singular if and only if* $s \in \underline{S}$.

PROOF. Assume $q-q'-1=0$. Then for each $s \in \{1,\dots,p'-\ell\}$ we have $A_1^{p-p'+\ell+s} = -s-1$. By (5.5.3) we conclude that $\lambda_{p-p'+\ell+s}$ is $\Delta^+(\mathfrak{k})$ dominant integral. This proves (1).

Suppose now that $q-q'-1>0$ and $s \in \{1,\dots,p'-\ell\}$ lies in $\underline{S}$. Then $s=s_t$ for some $t \in \{0,\dots,q-q'-2\}$. One has

$$A_{(q-q'-1)-t}^{p-p'+\ell+s_t} = -\Big(s_t - t - \sum_{i=n-q'-t}^{n-q'} a_i\Big).$$

But this is zero by the definition of s_t. Hence by (5.5.2) we conclude that $\lambda_{p-p'+\ell+s_t}$ is $\mathfrak{k}$ singular. This proves (2).

Suppose $q-q'-1>0$ and $s \in \{1,\dots,p'-\ell\}$ with $s_t < s < s_{t+1}$ for $s_t, s_{t+1} \in \underline{S}$. We have $s_t+1 \le s \le s_{t+1}-1$, i.e.,

$$t+1+\sum_{i=n-q'-t}^{n-q'} a_i \le s \le t+\sum_{i=n-q'-t-1}^{n-q'} a_i. \tag{5.10}$$

We find by the left side of (5.10) that

$$A_{q-q'-1-t}^{p-p'+\ell+s} = t-s+\sum_{i=n-q'-t}^{n-q'} a_i \le -1$$

and by the right side of (5.10) that

$$A_{q-q'-1-(t+1)}^{p-p'+\ell+s} = t+1-s+\sum_{i=n-q'-t-1}^{n-q'} a_i \ge 1.$$

Thus by (5.6.2) $\lambda_{p-p'+\ell+s}$ is $\mathfrak{k}$ nonsingular and $\nu_{p-p'+\ell+s} = q-q'-t-2$. Furthermore, $\sigma_{n-q'-t-2}\cdots\sigma_{p+1}(\lambda_{p-p'+\ell+s}+\rho_k)-\rho_k$ is $\Delta^+(\mathfrak{k})$ dominant integral by (5.5.4). This proves the first part of (3). We now suppose that $s \in \{1,\dots,p'-\ell\}$ satisfies $s > s_{q-q'-2} = \max(\underline{S})$. One then finds $A_1^{p-p'+\ell+s} = -(s-s_{q-q'-2}) < 0$ so that by (5.5.3) we conclude that $\lambda_{p-p'+\ell+s}$ is $\Delta^+(\mathfrak{k})$ dominant integral. This completes the proof of (3). Finally, we observe that (4) follows immediatedly from (2) and (3).

We summarize (5.9) by plotting the pairs (j,ν) such that $\sigma_{p+\nu}\cdots\sigma_{p+1}(\lambda_j+\rho_k)-\rho_k$ is $\Delta^+(\mathfrak{k})$ dominant integral. For brevity, put

$$J = p-p'+\ell \text{ and } N = q-q'-1. \tag{5.11}$$

If the set $B = \{j : 0 \leq j \leq J-1, \lambda_j \text{ is } \mathfrak{k} \text{ nonsingular}\}$ is nonempty, we put $j_0 = \max(B)$. Observe that for any j satisfying $0 \leq j \leq J-1$ one has $A_1^j \geq 0$ so that by (5.5.3) each λ_j with $0 \leq j \leq J-1$ is not $\Delta^+(\mathfrak{k})$ dominant. If λ_j with $0 \leq j \leq J-1$ is $\mathfrak{k}$ nonsingular, then the integer ν_j must be $q-1$ by (5.8.1). Consequently, $A_{q-1}^j > 0$ and so by (5.6.1) it follows that if $B \neq \varnothing$, then $\nu_j = q-1$ for all $0 \leq j \leq j_0$. We include these observations in the following tables by assuming $B \neq \varnothing$ and indicating j_0. Table 5.12.1 (resp. 5.12.2) corresponds to the case $N = 0$ (resp. $N > 0$).

Table 5.12.1

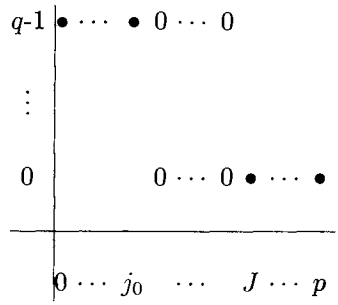
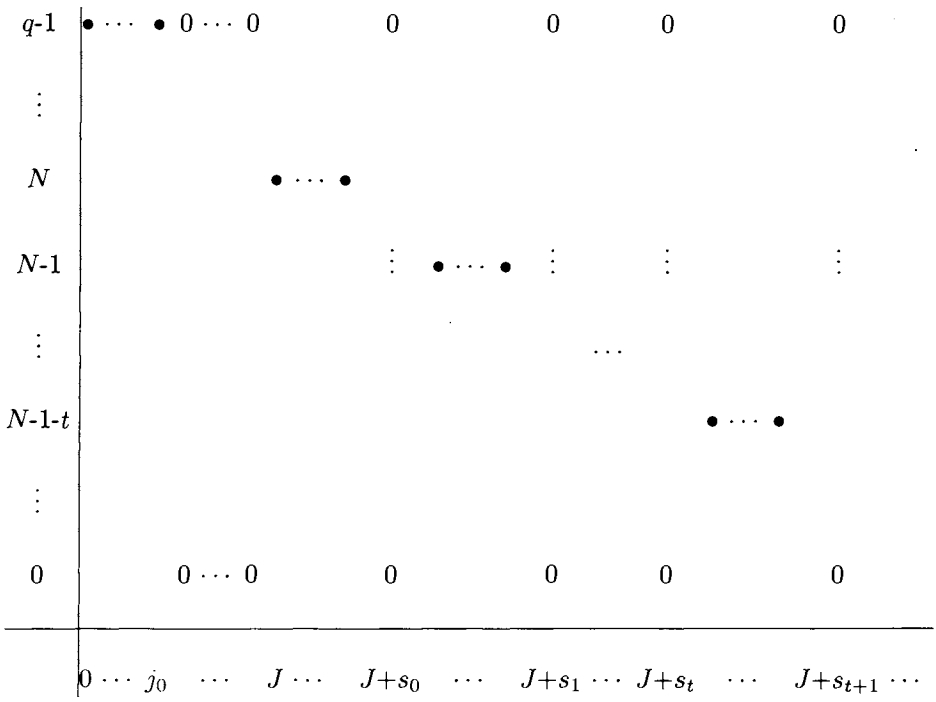


Table 5.12.2



Put $E_1^{i,j} = \Gamma^j(\mathrm{pro}_{\mathfrak{q}_1,M}^{\mathfrak{g},M}(F_M(\lambda_i)))$. The E_2 terms of (4.23) are obtained by computing cohomology down the rows of (5.12). More precisely, if $d_1^{ij} : E_1^{ij} \to E_1^{i+1,j}$ denotes the differential of bidegree $(1,0)$, then one has $E_2^{i,j} = \mathrm{Ker}(d_1^{i,j})/\mathrm{im}(d_1^{i-1,j})$. Similarly, the E_{r+1} term of the spectral sequence is determined by the differentials $d_r^{ij} : E_r^{ij} \to E_r^{i+r,j-r+1}$ (of bidegree $(r, 1-r)$) and is defined to be

$$E_{r+1}^{i,j} = \mathrm{Ker}(d_r^{i,j})/\mathrm{im}(d_r^{i-r,j+r-1}). \tag{5.13}$$

We now find certain E_∞ terms. Set $I = N + J$, where N and J are in (5.11). In order to count the number of columns between the J^{th} column and the next nontrivial column, set $\Sigma = \{s > 0 : \lambda_{J+s} \text{ is } \mathfrak{k} \text{ nonsingular}\}$ and define the integer $s_* \in \{0, \dots, p-J\}$ to be

$$s_* = \begin{cases} \min \Sigma & \text{if } \Sigma \neq \varnothing, \\ 0 & \text{if } \Sigma = \varnothing. \end{cases} \tag{5.14}$$

(5.15) PROPOSITION. *Assume that* $I < q-1$.

(1) $E_\infty^{u,I-u} = 0$ *if* $u \neq J$.

(2) *If* $s_* = 0$, *then* $E_1^{J,N} = \cdots = E_\infty^{J,N}$. *If* $s_* \geq 1$, *then* $E_1^{J,N} = \cdots = E_{s_*}^{J,N}$ *and* $E_\infty^{J,N} = E_{s_*+1}^{J,N} = \mathrm{Ker}(d_{s_*}^{J,N})$.

(3) *If* $s_* \geq 2$, *then there exists a chain*

$$E_{s_*}^{J+s_*,N-s_*+1} = \mathrm{Ker}(d_{s_*-1}^{J+s_*,N-s_*+1}) \subset \mathrm{Ker}(d_{s_*-2}^{J+s_*,N-s_*+1}) \subset \cdots \subset \mathrm{Ker}(d_1^{J+s_*,N-s_*+1}) \subset E_1^{J+s_*,N-s_*+1}.$$

PROOF. Assume $I < q-1$. It is clear by (5.12) that the only nonzero entry on or below the line $y = -x + I$ occurs at the point (J, N). In particular, this implies that $E_2^{u,I-u} = 0$ if $u \neq J$ from which one concludes $E_\infty^{u,I-u} = 0$ if $u \neq J$. This proves (1).

To prove (2), recall by (5.13) that $E_{r+1}^{i,j} = \mathrm{Ker}(d_r^{i,j})/\mathrm{im}(d_r^{i-r,j+r-1})$ where the differential $d_r^{i-r,j+r-1}$ maps $E_r^{i-r,j+r-1}$ to $E_r^{i,j}$. Since the point $(J-r, N+r-1)$, $r \geq 1$, lies below the line $y = -x + I$, it follows that $E_2^{J-r,N+r-1} = 0$ for $r \geq 1$. Since $E_r^{J-r,N+r-1}$ is a subquotient of $E_2^{J-r,N+r-1}$, we therefore have $\mathrm{im}(d_r^{J-r,N+r-1}) = 0$ for $r \geq 1$. Consequently, $E_{r+1}^{J,N} = \mathrm{Ker}(d_r^{J,N})$ for $r \geq 1$.

We now consider the differentials $d_r^{J,N} : E_r^{J,N} \to E_r^{J+r,N-r+1}$ and the spaces $E_r^{J+r,N-r+1}$ for $r \geq 1$. Let us first consider the case where $s_* \geq 1$. We claim that the form of (5.12) implies that there exists only one possible nontrivial $E_2^{J+r,N-r+1}$ for $r \geq 1$ and that is when $r = s_*$. For the case $N = 0$, (5.12.1) makes this obvious. For the case $N > 0$, one has $s_* - 1$ zero columns following the J^{th} column in (5.12.2). The only nontrivial entry in column $J+s_*$ has coordinates $(J + s_*, N - s_* + 1)$. Consequently, we have $E_2^{J+r,N-r+1} = 0$ for $1 \leq r < s_*$. We also have $E_2^{J+r,N-r+1} = 0$ for $r > s_*$ by (5.12.2). Thus each of the maps $d_r^{J,N} : E_r^{J,N} \to E_r^{J+r,N-r+1}$, $r \neq s_*$, is trivial since $E_r^{J+r,N-r+1}$ is a subquotient of $E_2^{J+r,N-r+1}$. Since $d_r^{J,N} = 0$ for $1 \leq r < s_*$, we have $E_1^{J,N} = \cdots = E_{s_*}^{J,N}$

and since $d_r^{J,N} = 0$, for $r > s_*$, we have $\operatorname{Ker}(d_{s_*}^{J,N}) = E_{s_*+1}^{J,N} = \cdots = E_\infty^{J,N}$. This proves (2) in the case where $s_* \geq 1$. If $s_* = 0$, then by definition of s_*, there are only zero columns following the J^{th} column. Thus $d_r^{J,N} = 0$ for $r > 0$ so that $E_1^{J,N} = \cdots = E_\infty^{J,N}$. This completes the proof of (2).

To prove (3), we assume $s_* \geq 2$. By definition of s_*, there exist $s_* - 1$ zero columns between column J and column $J + s_*$. In particular, we have $E_2^{J+i,N-i} = 0$ for $1 \leq i \leq s_* - 1$. Since $d_{s_*-i}^{J+i,N-i}$ is defined on $E_{s_*-i}^{J+i,N-i}$, a subquotient of $E_2^{J+i,N-i}$, it follows that $d_{s_*-i}^{J+i,N-i} = 0$ for $1 \leq i \leq s_* - 1$. But then we have

(5.16)
$$E_{s_*-i+1}^{J+s_*,N-s_*+1} = \operatorname{Ker}(d_{s_*-i}^{J+s_*,N-s_*+1})/\operatorname{im}(d_{s_*-i}^{J+i,N-i}) = \operatorname{Ker}(d_{s_*-i}^{J+s_*,N-s_*+1}),$$

for $1 \leq i \leq s_* - 1$. For $i = 1$, we have $E_{s_*}^{J+s_*,N-s_*+1} = \operatorname{Ker}(d_{s_*-1}^{J+s_*,N-s_*+1})$. But the map $d_{s_*-1}^{J+s_*,N-s_*+1}$ is defined on $E_{s_*-1}^{J+s_*,N-s_*+1} = \operatorname{Ker}(d_{s_*-2}^{J+s_*,N-s_*+1})$ so that $\operatorname{Ker}(d_{s_*-1}^{J+s_*,N-s_*+1}) \subset \operatorname{Ker}(d_{s_*-2}^{J+s_*,N-s_*+1})$. Using (5.16), one continues in this way to establish (3). This completes the proof.

(5.17) COROLLARY. *Assume that $I < q-1$ and $s_* \geq 1$. Then the differential $d_{s_*}^{J,N}$ maps $E_1^{J,N}$ to $E_1^{J+s_*,N-s_*+1}$.*

PROOF. The case $s_* = 1$ is clear since $d_1^{J,N}$ is of bidegree $(1,0)$. Consider the case $s_* > 1$. By (5.15.2) we have $E_1^{J,N} = E_{s_*}^{J,N}$. By (5.15.3) we have $E_{s_*}^{J+s_*,N-s_*+1} \subset E_1^{J+s_*,N-s_*+1}$. Since $d_{s_*}^{J,N}$ maps $E_{s_*}^{J,N}$ to $E_{s_*}^{J+s_*,N-s_*+1}$, the corollary follows.

We end this section by describing the connection between the integer s_* in (5.14) and the weight λ in (5.7).

(5.18) PROPOSITION. *Let λ in (5.7) be a reduction point and suppose $s_* \geq 1$.*

(1) *If $N = 0$, then $s_* = 1$.*

(2) *Assume $N > 0$. Then $s_* = 1$ if and only if $a_{n-q'} > 1$. If $a_{n-q'} = 1$, then $s_* \leq N + 1$ and s_* is given by*

$$s_* = \begin{cases} \min(Q) + 1 & \text{if } Q \neq \varnothing, \\ N + 1 & \text{if } Q = \varnothing, \end{cases}$$

where $Q = \{i : 1 \leq i \leq N - 1 \text{ and } a_{n-q'-i} > 0\}$.

PROOF. If $N = 0$, then by (5.9.1) we know λ_{J+1} is $\Delta^+(\mathfrak{k})$ dominant integral and is $\mathfrak{k}$ nonsingular. By the definition of s_* we conclude $s_* = 1$. This proves (1).

Now assume $N > 0$. By (5.9.4), the first zero column following the J^{th} column is $J + s_0$. Therefore $s_* = 1$ if and only if $s_0 > 1$. But $s_0 = a_{n-q'}$. This proves the first part of (2).

Now suppose that $a_{n-q'} = 1$. Recall that columns $J+1, \ldots, J+s_*-1$ are zero. By (5.9.4) the zero columns to the right of column J are determined by the set $\underline{S} = \{s_t : 0 \leq t \leq N-1\}$. Consequently, $s_* - 1 \leq N$ and $s_t = t + 1$ for

$0 \leq t \leq s_* - 2$. Since $a_{n-q'} = 1$ and $s_t = t + \sum_{i=n-q'-t}^{n-q'} a_i$ for $0 \leq t \leq N-1$, one has $\sum_{i=n-q'-t}^{n-q'-1} a_i = 0$ for $0 < t \leq s_* - 2$. Since $a_i \geq 0$ for $i \neq p+1$, we have

$$a_i = 0 \text{ for } n - q' - s_* + 2 \leq i \leq n - q' - 1. \tag{5.19}$$

It follows by (5.19) that $s_* = N+1$ iff $Q = \varnothing$. To prove (2), suppose $Q \neq \varnothing$. Then $s_* \leq N$ and by the above discussion one has $\underline{S} = \{1, 2, \ldots, s_* - 1, s_{(s_*-1)}, \ldots, s_{N-1}\}$. Since column $J + s_*$ is nonzero, $s_* \notin \underline{S}$ so that $s_* < s_{(s_*-1)}$. By (5.19) this inequality is equivalent to $a_{n-q'-s_*+1} > 0$. Thus $s_* - 1 = \min(Q)$. This completes the proof of (2).

6. Identifying covariant differential operators

The purpose of the present section is to show that the differential $d_{s_*}^{J,N}$ in (5.15) is a multiple of a distinguished covariant differential operator (CDO). In general, CDOs are G invariant differential operators defined on representation spaces of holomorphically induced representations of a Hermitian symmetric pair (G, K). We refer to [6] for the explicit form of such operators in standard realizations of the induced representation spaces. By [6], a special property of certain nonzero CDOs is that the $\mathfrak{k}$ finite vectors in their kernels form irreducible unitarizable $(\mathfrak{g}, K)$ modules. By utilizing this property, cohomology can be realized as irreducible unitarizable $(\mathfrak{g}, K)$ modules. We begin with a brief discussion of CDOs.

Suppose U_τ and U_μ are holomorphically induced representations of a Hermitian symmetric pair (G, K). A covariant differential operator $D_{\tau,\mu}$ is a continuous operator intertwining U_τ and U_μ. These operators arise as dual maps to particular $\mathfrak{g}$ maps between generalized Verma modules. It is known from the work of Boe, Collingwood, Enright and Shelton (cf. [2], [3]) that the spaces of such $\mathfrak{g}$ maps are at most one-dimensional. By duality, the same is true of CDOs. Thus a CDO is determined, up to a constant multiple, by the inducing parameters τ and μ.

We are interested only in those CDOs that realize irreducible unitary $(\mathfrak{g}, K)$ modules as their kernels. Such CDOs arise in the context of unitary highest weight representations. More precisely, if τ is a reduction point in the Enright, Howe and Wallach classification [8] and if $\mu = \mu(\tau)$ is the highest weight of the maximal submodule of $U(\mathfrak{g}) \otimes_{U(\mathfrak{k}\oplus\mathfrak{p}_+)} F_K(\tau)$, then the space of K finite vectors in $\mathrm{Ker}(D_{\tau,\mu})$ is an irreducible unitary highest weight $(\mathfrak{g}, K)$ module (cf. (3.2) [6]). All pairs of weights (τ, μ) for which $D_{\tau,\mu}$ has this property can be found in section six of [5]. For convenience we give such relevant pairs below. Before doing so, we recall the cone decomposition of the set Λ_τ of reduction points. By (6.6) [5], Λ_τ is the disjoint union of finitely many cones. A cone C in this decomposition is defined by the condition that $\tau - \mu(\tau)$ is constant for all τ in C. In the $SU(p,q)$ case, each cone can be parametrized by a triple (a, b, m) of positive integers. Here a and b measure a kind of singularity in the weight τ and m is the degree of the CDO $D_{\tau,\mu}$. We summarize the results of (6.21) and

(6.22) in [5] for the group $SU(p,q)$ and give the relevant CDO pairs (τ,μ) along with the associated cone data (a,b,m). Recall that $n=p+q$.

(6.1) DEFINITION. We say that (τ,μ) is a CDO pair (for $SU(p,q)$) with cone data (a,b,m) if: τ and μ are $\Delta^+(\mathfrak{k})$ dominant integral weights of the form

$$\tau=\sum_{i=a}^{p-1}c_i\omega_i-\Big(n-a-b+m-1+\sum_{i\neq p}c_i\Big)\omega_p+\sum_{i=p+1}^{n-b}c_i\omega_i,$$

and

$$\mu=\omega_{a-m}+(c_a-1)\omega_a+\sum_{i=a+1}^{p-1}c_i\omega_i-\Big(n-a-b+m-1+\sum_{i\neq p}c_i\Big)\omega_p+\sum_{i=p+1}^{n-b-1}c_i\omega_i+(c_{n-b}-1)\omega_{n-b}+\omega_{n-b+m}$$

where $a\in\{1,\dots,p\}$ and $b\in\{1,\dots,q\}$ are defined by

$$a=\begin{cases}\min\{i:i\in\{1,\dots,p-1\},c_i>0\} & \text{if } \exists\ i\in\{1,\dots,p-1\} \text{ with } c_i>0,\\ p & \text{if } c_1=\cdots=c_{p-1}=0,\end{cases}$$

$$b=\begin{cases}\min\{i:i\in\{1,\dots,q-1\},c_{n-i}>0\} & \text{if } \exists\ i\in\{1,\dots,q-1\} \text{ with } c_{n-i}>0,\\ q & \text{if } c_{p+1}=\cdots=c_{n-1}=0,\end{cases}$$

and m is an integer satisfying $1\le m\le\min\{a,b\}$. In the case $a=p$, then the term $(c_a-1)\omega_a$ appearing in μ is replaced with $-\omega_p$ and similarly, if $b=q$, then the term $(c_{n-b}-1)\omega_{n-b}$ in μ is replaced with $-\omega_p$.

We resume the notation of section five. In particular, recall the form of λ in (5.7), the definition of the integer s_* in (5.14) and integers J and N in (5.11). Note that if $s_*\ge 1$ then $\ell<p'$ and by (5.18) we have $s_*\le N+1$.

(6.2) PROPOSITION. *Let λ in (5.7) be a reduction point and assume $s_*\ge 1$. Put*

$$\tau=\begin{cases}\lambda_J & \text{if } N=0,\\ \sigma_{p+N}\cdots\sigma_{p+1}(\lambda_J+\rho_k)-\rho_k & \text{if } N>0,\end{cases}$$

$$\mu=\begin{cases}\lambda_{J+1} & \text{if } N=0,\\ \lambda_{J+N+1} & \text{if } N>0 \text{ and } s_*=N+1,\\ \sigma_{p+N-s_*+1}\cdots\sigma_{p+1}(\lambda_{J+s_*}+\rho_k)-\rho_k & \text{if } N>0 \text{ and } s_*<N+1.\end{cases}$$

Then (τ,μ) is a CDO pair with cone data $(p'-\ell,q'+s_*,s_*)$.

PROOF. First consider the case $N=0$. By (5.18.1) we have $s_*=1$. Now by (5.8.2) λ_J is $\Delta^+(\mathfrak{k})$ dominant integral and by (5.9.1) λ_{J+1} is $\Delta^+(\mathfrak{k})$ dominant integral. Moreover, by (5.3) we have

$$\lambda_J=\omega_{p'-\ell}+\sum_{i=p'-\ell}^{p-1}a_{i+1}\omega_i-\Big(J+1+\sum_{i=p'-\ell+1}^{p}a_i\Big)\omega_p$$

and

$$\lambda_{J+1} = \omega_{p'-\ell-1} + \sum_{i=p'-\ell-1}^{p-1} a_{i+1}\omega_i - \big(J+2+\sum_{i=p'-\ell}^{p} a_i\big)\omega_p + \omega_{p+1}.$$

Now $a_{p'-\ell} = 0$ so we see by (6.1) that $(\lambda_J, \lambda_{J+1})$ is a CDO pair with cone data $(p'-\ell, q, 1)$. Since $N = 0$, we have $q = q'+1 = q'+s_*$. This proves the proposition for $N = 0$.

Now consider the case $N > 0$. By (5.8.2) we know that $\tau = \sigma_{p+N}\cdots\sigma_{p+1}(\lambda_J + \rho_k) - \rho_k$ is $\Delta^+(\mathfrak{k})$ dominant integral. By the proof of (5.5.4) with $\nu_J = N$ we have

$$\tau = \omega_{p'-\ell} + \sum_{i=p'-\ell}^{p+N-1} a_{i+1}\omega_i + (A_N^J - 1)\omega_{p+N} - (A_{N+1}^J + 1)\omega_{p+N+1}. \tag{6.3}$$

By the proof of (5.8.2) we know $A_N^J = a_{n-q'}$ and $A_{N+1}^J = -1$. In order to identify the precise form of the weight τ in (6.3), we must consider the cases $a_{n-q'} > 1$ and $a_{n-q'} = 1$ separately.

Suppose first that $a_{n-q'} > 1$. Then we have $\tau = \omega_{p'-\ell} + \sum_{i=p'-\ell}^{p+N-1} a_{i+1}\omega_i + (a_{n-q'}-1)\omega_{p+N}$. Now by (5.18.2) we have $s_* = 1$. Set $\mu = \sigma_{p+N}\cdots\sigma_{p+1}(\lambda_{J+1} + \rho_k) - \rho_k$. By the proof of (5.5.4) we know μ takes the form

$$\mu = \omega_{p'-\ell-1} + \sum_{i=p'-\ell-1}^{p+N-1} a_{i+1}\omega_i + (A_N^{J+1} - 1)\omega_{p+N} - (A_{N+1}^{J+1} + 1)\omega_{p+N+1}.$$

But $A_N^{J+1} = a_{n-q'} - 1$ and $A_{N+1}^{J+1} = -2$. We see that μ is $\Delta^+(\mathfrak{k})$ dominant integral. Moreover, by (6.1) it follows that (τ, μ) is a CDO pair with cone data $(p'-\ell, q'+1, 1) = (p'-\ell, q'+s_*, s_*)$. This proves the proposition in the case $a_{n-q'} > 1$.

We now suppose $a_{n-q'} = 1$. By (5.18.2) we have $1 < s_* \leq N+1$. According to (5.18.2), we need to consider the cases $s_* = N+1$ and $s_* < N+1$ separately. We assume first that $s_* = N+1$. Then the set Q in (5.18.2) is empty so that $a_{p+2} = \cdots = a_{n-q'-1} = 0$. Then τ in (6.3) is

$$\tau = \omega_{p'-\ell} + \sum_{i=p'-\ell}^{p} a_{i+1}\omega_i = \omega_{p'-\ell} + \sum_{i=p'-1}^{p} a_{i+1}\omega_i.$$

By (5.3) we know $\mu = \lambda_{J+N+1}$ has the form

$$\omega_{p-(J+N+1)} + \sum_{i=p-(J+N+1)}^{p-1} a_{i+1}\omega_i - \Big(J+N+2+\sum_{i=p-(J+N)}^{p} a_i\Big)\omega_p$$
$$+ \Big(J+N+1+\sum_{i=p-(J+N)}^{p+1} a_i\Big)\omega_{p+1} + \omega_{n-q'} = \omega_{p'-\ell-s_*} + \sum_{i=p'-1}^{p-1} a_{i+1}\omega_i$$
$$- \Big(J+N+2+\sum_{i=p'}^{p} a_i\Big)\omega_p + \Big(J+N+1+\sum_{i=p'}^{p+1} a_i\Big)\omega_{p+1} + \omega_{n-q'}.$$

Since $-a_{p+1} = J+N+1+\sum_{i=p'}^{p} a_i$ and $p+s_* = n-q'$, it now follows by (6.1) that (τ,μ) is a CDO pair with cone data $(p'-\ell, q, s_*) = (p'-\ell, q'+s_*, s_*)$.

We now consider the case $a_{n-q'} = 1$ and $1 < s_* < N+1$. Then by (5.18.2) we have $0 = a_{n-q'-1} = \cdots = a_{n-q'-s_*+2}$ and $a_{n-q'-s_*+1} > 0$. From (6.3) we see $\tau = \sigma_{p+N}\cdots\sigma_{p+1}(\lambda_J+\rho_k) - \rho_k$ has the form

$$\tau = \omega_{p'-\ell} + \sum_{i=p'-\ell}^{p+N-1} a_{i+1}\omega_i = \omega_{p'-\ell} + \sum_{i=p'-1}^{n-q'-s_*} a_{i+1}\omega_i.$$

With $\mu = \sigma_{p+N-s_*+1}\cdots\sigma_{p+1}(\lambda_{J+s_*}+\rho_k) - \rho_k$, we know by the proof of (5.5.4) that μ has the explicit form

$$\omega_{p'-\ell-s_*} + \sum_{i=p'-1}^{n-q'-s_*-1} a_{i+1}\omega_i + (A^{J+s_*}_{N-s_*+1} - 1)\omega_{n-q'-s_*}$$
$$- (A^{J+s_*}_{N-s_*+2} + 1)\omega_{n-q'-s_*+1} + \omega_{n-q'}.$$

One finds that $A^{J+s_*}_{N-s_*+1} = a_{n-q'-s_*+1}$ and $A^{J+s_*}_{N-s_*+2} = -1$. It now follows from (6.1) that (τ,μ) is a CDO pair with cone data $(p'-\ell, q'+s_*, s_*)$. This completes the proof.

We now make clear the connection between the produced modules appearing in (4.26), the differential $d^{J,N}_{s_*}$ in (5.17) and the CDO pairs (τ,μ) of this section. We begin by recalling a well-known fact regarding produced modules. Let τ denote an analytically integral, $\Delta^+(\mathfrak{k})$ dominant weight and let $F_K(\tau)$ denote the representation space for the representation of K with highest weight τ. A standard realization for the holomorphically induced representation space corresponding to τ is the space $\mathcal{O}(\tau)$ of $F_K(\tau)$ valued holomorphic functions on G/K. The representation of G on this space is given by a multiplier representation (cf. (1.2) [5]). By (1.10) [5] the K finite vectors in $\mathcal{O}(\tau)$ form a $(\mathfrak{g},K)$ module equivalent to $\mathrm{pro}^{\mathfrak{g},K}_{\mathfrak{q}(\omega_p),K}(F_K(\tau))$. Consequently, by restricting to the space of K finite vectors, we may view the space of CDOs as a one-dimensional space $\mathcal{D}(\tau,\mu)$ of $(\mathfrak{g},K)$ module maps

$$D_{\tau,\mu} : \mathrm{pro}^{\mathfrak{g},K}_{\mathfrak{q}(\omega_p),K}(F_K(\tau)) \to \mathrm{pro}^{\mathfrak{g},K}_{\mathfrak{q}(\omega_p),K}(F_K(\mu)),$$

where (τ, μ) is a CDO pair.

Recall from section five that $I = J+N$ where $J = p-p'-\ell$ and $N = q-q'-1$.

(6.4) THEOREM. *Let λ in (5.7) be a reduction point. Assume $s_* \geq 1$ and $I < q-1$.*

(1) *There exists $\sigma \in \mathcal{W}$ with $\ell(\sigma) = I$ such that*

$$\mathrm{Ker}(d_{s_*}^{J,N}) = \Gamma^{\ell(\sigma)}(\mathrm{pro}_{\mathfrak{q},M}^{\mathfrak{g},M}(F_L(\lambda))).$$

(2) *The differential $d_{s_*}^{J,N}$ is a multiple of the CDO $D_{\tau,\mu}$ whose parameters are given by the CDO pairs (τ, μ) in (6.2).*

(3) *If $d_{s_*}^{J,N}$ is nonzero, $\Gamma^{\ell(\sigma)}(\mathrm{pro}_{\mathfrak{q},M}^{\mathfrak{g},M}(F_L(\lambda)))$ is an irreducible, unitarizable highest weight $(\mathfrak{g}, K)$ module with highest weight $\tau = \sigma(\lambda+\rho)-\rho$.*

PROOF. We begin by proving (2). Assume λ is a reduction point and $s_* \geq 1$. Then $\ell < p'$ and by (5.9) we know that (5.12) is a plotting of $E_1^{j,\nu} = \Gamma^{\nu}(\mathrm{pro}_{\mathfrak{q}_1,M}^{\mathfrak{g},M}(F_M(\lambda_j)))$, $0 \leq j, \nu$. If $I < q-1$, then by (5.17) the differential $d_{s_*}^{J,N}$ maps $E_1^{J,N}$ to $E_1^{J+s_*,N-s_*+1}$. But by (4.26) we have

$$E_1^{J,N} = \mathrm{pro}_{\mathfrak{q}(\omega_p),K}^{\mathfrak{g},K}(F_K(\sigma_{p+N}\cdots\sigma_{p+1}(\lambda_J+\rho_k)-\rho_k)) = \mathrm{pro}_{\mathfrak{q}(\omega_p),K}^{\mathfrak{g},K}(F_K(\tau))$$

and

$$\begin{aligned} E_1^{J+s_*,N-s_*+1} &= \mathrm{pro}_{\mathfrak{q}(\omega_p),K}^{\mathfrak{g},K}(F_K(\sigma_{p+N-s_*+1}\cdots\sigma_{p+1}(\lambda_{J+s_*}+\rho_k)-\rho_k)) \\ &= \mathrm{pro}_{\mathfrak{q}(\omega_p),K}^{\mathfrak{g},K}(F_K(\mu)), \end{aligned}$$

where (τ, μ) is clearly the CDO pair in (6.2). Thus $d_{s_*}^{J,N}$ lies in the one-dimensional space $\mathcal{D}(\tau, \mu)$. This proves (2).

We now prove (1). Since $s_* \geq 1$, we have by (5.15.2) that $E_\infty^{J,N} = \mathrm{Ker}(d_{s_*}^{J,N})$. By (5.15.1) we also know that $E_\infty^{u,I-u} = 0$ if $u \neq J$. Since $E_2^{n,m} \underset{n}{\Longrightarrow} \Gamma^{n+m}(\mathrm{pro}_{\mathfrak{q},M}^{\mathfrak{g},M}(F_L(\lambda)))$ by (4.23), it follows by (5.4), p.326 [4] that $E_\infty^{J,I-J} = \Gamma^I(\mathrm{pro}_{\mathfrak{q},M}^{\mathfrak{g},M}(F_L(\lambda)))$. This proves that $\mathrm{Ker}(d_{s_*}^{J,N}) = \Gamma^I(\mathrm{pro}_{\mathfrak{q},M}^{\mathfrak{g},M}(F_L(\lambda)))$. It remains to show that there exists $\sigma \in \mathcal{W}$ with $\ell(\sigma) = I$ and $\tau = \sigma(\lambda+\rho)-\rho$. Since $\rho-\rho_k$ is a multiple of ω_p and since the Weyl group element $\sigma_{p+N}\cdots\sigma_{p+1}$ fixes ω_p, it follows that

$$\tau = \sigma_{p+N}\cdots\sigma_{p+1}(\lambda_J+\rho_k)-\rho_k = \sigma_{p+N}\cdots\sigma_{p+1}(\lambda_j+\rho)-\rho.$$

But by (5.3) we have $\lambda_J = \sigma_p\cdots\sigma_{p-J+1}(\lambda+\rho)-\rho$. Consequently, by putting $\sigma = \sigma_{p+N}\cdots\sigma_{p-J+1}$ we then have $\tau = \sigma(\lambda+\rho)-\rho$ and $\ell(\sigma) = J+N = I$. This proves (1).

To prove (3), assume $d_{s_*}^{J,N}$ is nonzero. By (2), this differential is a distinguished nonzero CDO and so by (3.2) [6] $\mathrm{Ker}(d_{s_*}^{J,N})$ is an irreducible unitary highest weight $(\mathfrak{g}, K)$ module with highest weight τ. Then (3) follows by (1).

(6.5) COROLLARY. *With the assumptions and notation of (6.4), the* $(\mathfrak{g}, K)$ *module* $\Gamma^{\ell(\sigma)}(\mathrm{pro}^{\mathfrak{g},M}_{\mathfrak{q},M}(F_L(\lambda)))$ *is irreducible if it is a proper submodule of* $\mathrm{pro}^{\mathfrak{g},K}_{\mathfrak{q}(\omega_p),K}(F_K(\tau))$.

PROOF. If $\Gamma^{\ell(\sigma)}(\mathrm{pro}^{\mathfrak{g},M}_{\mathfrak{q},M}(F_L(\lambda)))$ is a proper submodule, then the differential $d^{J,N}_{s_*}$ is not zero by (6.4.1). But then by (6.4.3), it follows that this submodule is irreducible.

7. An irreducibility condition

This section provides a sufficient condition for which the $(\mathfrak{g}, K)$ module $\Gamma^{\ell(\sigma)}(\mathrm{pro}^{\mathfrak{g},M}_{\mathfrak{q},M}(F_L(\lambda)))$ with λ as in (6.4) is irreducible. By (6.5), it suffices to show that this submodule is proper in $\mathrm{pro}^{\mathfrak{g},K}_{\mathfrak{q}(\omega_p),K}(F_K(\tau))$. We turn to multiplicity results to identify conditions under which the K type $F_K(\mu)$ of $\mathrm{pro}^{\mathfrak{g},K}_{\mathfrak{q}(\omega_p),K}(F_K(\tau))$ is not contained in $\Gamma^{\ell(\sigma)}(\mathrm{pro}^{\mathfrak{g},M}_{\mathfrak{q},M}(F_L(\lambda)))$. We begin by considering a spectral sequence that determines the multiplicity of K types in derived functor modules.

(7.1) THEOREM. *Let* W *be an* (ℓ, M) *module. Choose a positive system of roots* Δ^+_k *of* $\mathfrak{k}$ *which is compatible with the parabolic* $\overline{\mathfrak{q}} \cap \mathfrak{k}$. *Let* (π, Z) *be an irreducible representation of* K. *Put* $S = \dim(\mathfrak{u} \cap \mathfrak{k})$ *and* $D_1 = \Lambda^{\dim(\overline{\mathfrak{u}} \cap \mathfrak{k})}(\overline{\mathfrak{u}} \cap \mathfrak{k})$. *Then there exists a spectral sequence*

$$\bigoplus_{m=0}^{r} \mathrm{Hom}_M(H^{S-i}(\overline{\mathfrak{u}} \cap \mathfrak{k}, Z) \otimes D_1, S^m(\overline{\mathfrak{u}} \cap \mathfrak{p}) \otimes W) \Rightarrow \mathrm{Ext}^i_{\mathfrak{k},M}(Z, \mathrm{pro}^{\mathfrak{g},M}_{\mathfrak{q},M}(W)).$$

Here the integer r *is fixed but depends on* W *and* Z.

PROOF. This is the spectral sequence appearing in the proof of (6.3.12) [18]. Here we have replaced the parabolic $\mathfrak{q}$ by $\overline{\mathfrak{q}}$ and dropped the tensoring of W with $D = \Lambda^{\dim(\mathfrak{u})}\mathfrak{u}$. Keeping in mind these minor changes, we find (6.3.20) in [18] is replaced by the above spectral sequence.

By (6.2.15) [18], the multiplicity of (π, Z) in $\Gamma^i(\mathrm{pro}^{\mathfrak{g},M}_{\mathfrak{q},M}(W))$ is given by the dimension of $\mathrm{Ext}^i_{\mathfrak{k},M}(Z, \mathrm{pro}^{\mathfrak{g},M}_{\mathfrak{q},M}(W))$. Since the above E_2 terms determine multiplicity of K types in cohomology, we proceed to write these terms in a convenient form.

We now apply Kostant's theorem ((3.2.16) [18]) to write $H^{S-i}(\overline{\mathfrak{u}} \cap \mathfrak{k}, Z) \otimes D_1$ as the direct sum of M types. The relevant parabolic of $\mathfrak{k}$ is $\overline{\mathfrak{q}} \cap \mathfrak{k} = \mathfrak{m} \oplus (\overline{\mathfrak{u}} \cap \mathfrak{k})$ so that we need to choose a positive root system of $\mathfrak{k}$ containing the roots of $\overline{\mathfrak{u}} \cap \mathfrak{k}$. We make this choice however so that the positive roots $\Delta^+(\mathfrak{m})$ of $\mathfrak{m}$ remain unchanged. Recall from section two our choice of positive roots Δ^+ of $\mathfrak{g}$. With $\alpha_i = e_i - e_{i+1}, 1 \leq i \leq n-1$, the set of simple roots corresponding to Δ^+ is $\mathcal{B}_{\mathfrak{g}} = \{\alpha_i : 1 \leq i \leq n-1\}$. The set of simple roots corresponding to $\Delta^+(\mathfrak{k})$ and $\Delta^+(\mathfrak{m})$ are $\mathcal{B}_k = \{\alpha_i : i \neq p\}$ and $\mathcal{S}(\mathfrak{m}) = \{\alpha_i : i \neq p, p+1\}$, respectively.

We put $\omega_0 = \sigma_{p+1} \cdots \sigma_{n-1} \in \mathcal{W}_k$ and consider a second choice $\omega_0(\Delta^+(\mathfrak{k}))$ of positive roots of $\mathfrak{k}$. Let ℓ (resp. ℓ_0) denote the length function for $\Delta^+(\mathfrak{k})$ (resp.

$\omega_0(\Delta^+(\mathfrak{k})))$. Let $\Delta^+(\mathfrak{u}\cap\mathfrak{k})$ denote the positive roots of $\mathfrak{u}\cap\mathfrak{k}$. Let $\rho_k, \rho_{k,0}$ and $\rho(\mathfrak{u}\cap\mathfrak{k})$ denote half the sum of roots $\Delta^+(\mathfrak{k}), \omega_0(\Delta^+(\mathfrak{k}))$ and $\Delta^+(\mathfrak{u}\cap\mathfrak{k})$, respectively. Put

$$\mathcal{W}_k^1 = \{\omega \in \mathcal{W}_k : \Delta^+(\mathfrak{m}) \subset \omega\Delta^+(\mathfrak{k})\}$$

and

$$\mathcal{W}_{k,0}^1 = \{\omega \in \mathcal{W}_k : \Delta^+(\mathfrak{m}) \subset \omega\omega_0(\Delta^+(\mathfrak{k}))\}.$$

(7.2) PROPOSITION. *With the above notation we have*

(1) $\omega_0(\Delta^+(\mathfrak{k})) = \Delta^+(\mathfrak{m}) \cup (-\Delta^+(\mathfrak{u}\cap\mathfrak{k}))$.
(2) $\rho_{k,0} = \rho_k - 2\rho(\mathfrak{u}\cap\mathfrak{k})$.
(3) $(\mathcal{W}_{k,0}^1)\omega_0 = \mathcal{W}_k^1$.
(4) $\ell_0(\omega) + \ell(\omega\omega_0) = S$ *for* $\omega \in \mathcal{W}_{k,0}^1$.

PROOF. One readily checks that the simple roots $\omega_0(\mathcal{B}_k)$ for $\omega_0(\Delta^+(\mathfrak{k}))$ is the set

$$\omega_0(\mathcal{B}_k) = \{\alpha_1, \ldots, \alpha_{p-1}, \alpha_{p+2}, \ldots, \alpha_{n-1}, -(\alpha_{p+1} + \cdots + \alpha_{n-1})\}.$$

In particular, one has $\mathcal{S}(\mathfrak{m}) \subset \omega_0(\mathcal{B}_k)$ so that $\Delta^+(\mathfrak{m}) \subset \omega_0(\Delta^+(\mathfrak{k}))$. In addition, we see that $-\Delta(\mathfrak{u}\cap\mathfrak{k}) = \{-(\alpha_{p+1} + \cdots + \alpha_{p+j}) : 1 \le j \le q-1\}$ is also contained in $\omega_0(\Delta^+(\mathfrak{k}))$. This proves (1).

To prove (2), first note that $\rho_k = \rho_{\mathfrak{m}} + \rho(\mathfrak{u}\cap\mathfrak{k})$, where $\rho_{\mathfrak{m}}$ is half the sum of positive roots of $\mathfrak{m}$. By (1) we have $\rho_{k,0} = \rho_{\mathfrak{m}} - \rho(\mathfrak{u}\cap\mathfrak{k}) = \rho_{\mathfrak{m}} + \rho(\mathfrak{u}\cap\mathfrak{k}) - 2\rho(\mathfrak{u}\cap\mathfrak{k}) = \rho_k - 2\rho(\mathfrak{u}\cap\mathfrak{k})$. The proof of (3) follows directly from the definitions of $\mathcal{W}_k^1$ and $\mathcal{W}_{k,0}^1$.

To prove (4), for each $\omega \in \mathcal{W}_k$ define sets $\Delta_0^+(\omega) = \{\alpha \in \omega_0(\Delta^+(\mathfrak{k})) : \omega^{-1}\alpha \in -\omega_0(\Delta^+(\mathfrak{k}))\}$ and $\Delta^+(\omega) = \{\alpha \in \Delta^+(\mathfrak{k}) : \omega^{-1}\alpha \in -\Delta^+(\mathfrak{k})\}$. Then by definition of the length function $\ell_0, \ell_0(\omega) = \#(\Delta_0^+(\omega))$. Likewise, $\ell(\omega) = \#(\Delta^+(\omega))$. By (3.2.2) [18] we know $\omega \in \mathcal{W}_k$ lies in $\mathcal{W}_k^1$ if and only if $\Delta^+(\omega) \subset \Delta(\mathfrak{u}\cap\mathfrak{k})$. Thus for $\omega \in \mathcal{W}_k^1$

$$\ell(\omega) = \#\{\alpha \in \Delta(\mathfrak{u}\cap\mathfrak{k}) : \omega^{-1}\alpha \in -\Delta^+(\mathfrak{k})\}.$$

Similarly, $\omega \in \mathcal{W}_k$ lies in $\mathcal{W}_{k,0}^1$ if and only if $\Delta_0^+(\omega) \subset \Delta(\overline{\mathfrak{u}}\cap\mathfrak{k}) = -\Delta(\mathfrak{u}\cap\mathfrak{k})$ so that for $\omega \in \mathcal{W}_{k,0}^1$ one has $\ell_0(\omega) = \#\{\alpha \in -\Delta(\mathfrak{u}\cap\mathfrak{k}) : \omega^{-1}\alpha \in \omega_0(\Delta^+(\mathfrak{k}))\}$. Fix $\omega \in \mathcal{W}_{k,0}^1$. Clearly $\Delta(\mathfrak{u}\cap\mathfrak{k})$ is the disjoint union

$$\Delta(\mathfrak{u}\cap\mathfrak{k}) = \{\alpha \in \Delta(\mathfrak{u}\cap\mathfrak{k}) : \omega^{-1}\alpha \in \omega_0(\Delta^+(\mathfrak{k}))\} \cup \{\alpha \in \Delta(\mathfrak{u}\cap\mathfrak{k}) : \omega^{-1}\alpha \in -\omega_0(\Delta^+(\mathfrak{k}))\}.$$

Define $\chi : \Delta_0^+(\omega) \to \{\alpha \in \Delta(\mathfrak{u}\cap\mathfrak{k}) : \omega^{-1}\alpha \in -\omega_0(\Delta^+(\mathfrak{k}))\}$ by $\chi(\alpha) = -\alpha$. By part (3), $\omega\omega_0 \in \mathcal{W}_k^1$ so that $\{\alpha \in \Delta(\mathfrak{u}\cap\mathfrak{k}) : \omega^{-1}\alpha \in -\omega_0(\Delta^+(\mathfrak{k}))\} = \Delta^+(\omega\omega_0)$. Since χ is bijective, we have $\#(\Delta(\mathfrak{u}\cap\mathfrak{k})) = \#(\Delta_0^+(\omega)) + \#(\Delta^+(\omega\omega_0))$. We therefore have $S = \#\Delta(\mathfrak{u}\cap\mathfrak{k}) = \ell_0(\omega) + \ell(\omega\omega_0)$. This proves (4) and so completes the proof.

One readily finds that $\mathcal{W}_k^1 = \{1, \sigma_{p+1}, \sigma_{p+1}\sigma_{p+2}, \ldots, \sigma_{p+1}\cdots\sigma_{n-1}\}$. In particular, each element $\omega \in \mathcal{W}_k^1$ is uniquely determined by its length $\ell(\omega)$.

(7.3) PROPOSITION. *Let π be an irreducible representation of K on the complex space Z. Let μ be the highest weight of π with respect to $\Delta^+(\mathfrak{k})$. Put $D_1 = \Lambda^{\dim(\overline{\mathfrak{u}}\cap\mathfrak{k})}(\overline{\mathfrak{u}}\cap\mathfrak{k})$. If $0 \le i \le S$, then $H^{S-i}(\overline{\mathfrak{u}}\cap\mathfrak{k}, Z)\otimes D_1$ is an irreducible representation of M with highest weight $\sigma_{p+1}\cdots\sigma_{p+i}(\mu+\rho)-\rho$.*

PROOF. By (3.2.16) [18], $H^{S-i}(\overline{\mathfrak{u}}\cap\mathfrak{k}, Z)$ is the multiplicity free direct sum of irreducible representations of M whose highest weights are of the form $\omega(\omega_0(\mu)+\rho_{k,0})-\rho_{k,0}$ where $\ell_0(\omega) = S-i$. Also, D_1 is a one-dimensional representation of M with weight $-2\rho(\mathfrak{u}\cap\mathfrak{k})$. So $H^{S-i}(\overline{\mathfrak{u}}\cap\mathfrak{k}, Z)\otimes D_1$ is the multiplicity free direct sum of irreducible M types with highest weights of the form

$$\omega(\omega_0(\mu)+\rho_{k,0})-\rho_{k,0}-2\rho(\mathfrak{u}\cap\mathfrak{k})$$

where $\ell_0(\omega) = S-i$. Since $\omega_0(\rho_k) = \rho_{k,0}$, (7.2.2) implies these weights are of the form $\omega\omega_0(\mu+\rho_k)-\rho_k$ where $\ell_0(\omega) = S-i$. By (7.2.3) and (7.2.4) we know $\omega\omega_0 \in \mathcal{W}_k^1$ and $\ell(\omega\omega_0) = S-\ell_0(\omega) = i$. The only element in $\mathcal{W}_k^1$ of length i is $\sigma_{p+1}\cdots\sigma_{p+i}$. Consequently,

$$H^{S-i}(\overline{\mathfrak{u}}\cap\mathfrak{k}, Z)\otimes D_1 \tilde{=} F_M(\sigma_{p+1}\cdots\sigma_{p+i}(\mu+\rho_k)-\rho_k).$$

Finally, we observe that every element of $\mathcal{W}_k^1$ fixes the fundamental dominant weight ω_p. Since $\rho-\rho_k$ is a multiple of ω_p, it follows that for any $\omega \in \mathcal{W}_k^1$ we have $\omega(\mu+\rho_k)-\rho_k = \omega(\mu+\rho)-\rho$. This completes the proof of the proposition.

As discussed above, we wish to determine the multiplicity of μ in $\Gamma^{\ell(\sigma)}(\mathrm{pro}_{\mathfrak{q},M}^{\mathfrak{g},M}(F_L(\lambda)))$ where (τ,μ) is the CDO pair of (6.2). Since $\overline{\mathfrak{u}}\cap\mathfrak{p} = \overline{\mathfrak{u}}\cap\mathfrak{p}_-$, (7.3) shows that the relevant E_2 terms in (7.1) are direct sums of the spaces

$$\mathrm{Hom}_M(F_M(\sigma_{p+1}\cdots\sigma_{p+I}(\mu+\rho)-\rho), S^m(\overline{\mathfrak{u}}\cap\mathfrak{p}_-)\otimes F_L(\lambda)), \quad m \ge 0, \tag{7.4}$$

where $I = \ell(\sigma) = J+N$. Observe that if $F_L(\lambda) = F_M(\lambda)$ were an irreducible representation of M and if Hom in (7.4) were nonzero, then $\sigma_{p+1}\cdots\sigma_{p+I}(\mu+\rho)-\rho-\lambda$ would be a weight of $S^m(\overline{\mathfrak{u}}\cap\mathfrak{p}_-)$ for some $m \ge 0$. We now employ a result of Schmid [17] to show that, for certain λ, $S(\overline{\mathfrak{u}}\cap\mathfrak{p}_-)$ contains no weights of this form.

Assume $F_L(\lambda) = F_M(\lambda)$, i.e. that the restriction of the L action on the space $F_L(\lambda)$ to the subgroup M gives an irreducible representation of M. By the Branching Theorem for $U(p+1)$ (p.186, [21]), it follows that $F_L(\lambda) = F_M(\lambda)$ if and only if the coefficients $a_1,\ldots,a_p$ in (5.7) are all zero. Thus $F_L(\lambda) = F_M(\lambda)$ if and only if $p' = p+1$. With $p' = p+1$ the weight λ in (5.7) takes the explicit form

$$\lambda = -\Big(J+N+\sum_{i=p+2}^{n-q'} a_i\Big)\omega_{p+1} + \sum_{i=p+2}^{n-q'} a_i\omega_i \tag{7.5}$$

where $J = \ell-1$ and $N = q-q'-1$.

(7.6) LEMMA. *For $m \in \{1, \dots, q-1\}$ we have*

(1) $\sigma_{p+1} \cdots \sigma_{p+m}(\sum_{i=1}^{n-1} b_i\omega_i) = \sum_{i=1}^{p-1} b_i\omega_i + (\sum_{i=p}^{p+m} b_i)\omega_p - (\sum_{i=p+1}^{p+m} b_i)\omega_{p+1} + \sum_{i=p+2}^{p+m+1} b_{i-1}\omega_i + \sum_{i=p+m+1}^{n-1} b_i\omega_i$.

(2) $\sigma_{p+1} \cdots \sigma_{p+m}(\rho) - \rho = m\omega_p - (m+1)\omega_{p+1} + \omega_{p+m+1}$.

PROOF. Part (1) follows by induction on m and (2) follows immediately from (1).

(7.7) PROPOSITION. *Assume $p' = p+1$ and let (τ, μ) be the CDO pair in (6.2). Then we have*

$$(\sigma_{p+1} \cdots \sigma_{p+I}(\mu+\rho) - \rho) - \lambda = \begin{cases} \omega_{p-s_*} - 2\omega_p + \omega_{p+1} - \omega_{n-q'-s_*+1} + \omega_{n-q'} & \text{if } \ell = 1, \\ \omega_{p+1-\ell-s_*} - \omega_p - \omega_{n-q'-s_*+1} - \omega_{n-q'} + \omega_{n-q'+1} + \omega_{n-q'+\ell-1} & \text{if } \ell > 1. \end{cases}$$

PROOF. Set $\gamma = \sigma_{p+1} \cdots \sigma_{p+I}$. To identify the form of μ, we need to consider the three cases given in (6.2).

Assume $N = 0$. Then $s_* = 1$ by (5.18.1), $\lambda = -J\omega_{p+1}$ by (7.5) and $\mu = \lambda_{J+1}$ by (6.2). We first verify

$$\gamma(\mu) = \begin{cases} \omega_{p-1} - 2\omega_p + \omega_{p+1} & \text{if } J = 0, \\ \omega_{p-J-1} - (J+1)\omega_p - \omega_{p+1} + \omega_{p+2} & \text{if } J > 0. \end{cases} \tag{7.8}$$

If $J = 0$, then $\lambda = 0$ and $\gamma = 1$ so that $\gamma(\mu) = \lambda_1 = \sigma_p(\rho) - \rho$ by (5.2). But $\sigma_p(\rho) - \rho = -\alpha_p = \omega_{p-1} - 2\omega_p + \omega_{p+1}$. If $J > 0$ then by (5.3) $\lambda_{J+1} = \omega_{p-J-1} - (J+2)\omega_p + \omega_{p+1}$ so that by (7.6.1) $\gamma(\mu) = \omega_{p-J-1} - (J+1)\omega_p - \omega_{p+1} + \omega_{p+2}$. This proves (7.8). From (7.8) and (7.6.2) we conclude

$$(\sigma_{p+1} \cdots \sigma_{p+I}(\mu+\rho) - \rho) - \lambda = \begin{cases} \omega_{p-1} - 2\omega_p + \omega_{p+1} & \text{if } J = 0, \\ \omega_{p-J-1} - \omega_p - 2\omega_{p+1} + \omega_{p+2} + \omega_{p+J+1} & \text{if } J > 0. \end{cases} \tag{7.9}$$

Since $J = \ell - 1$ and $s_* = 1$, (7.9) agrees with the expression in the proposition and so establishes this case.

Assume $N > 0$ and $s_* = N+1$. By (5.18.2), $a_{n-q'} = 1$, and $a_{p+i} = 0$ for $2 \le i \le N$. Thus by (7.5) $\lambda = -(J+N+1)\omega_{p+1} + \omega_{n-q'}$. By (6.2) and (5.3), $\mu = \lambda_{J+N+1} = \omega_{p-\ell-N} - (J+N+2)\omega_p + \omega_{n-q'}$. By (7.6.1) we have

$$\gamma(\mu) = \begin{cases} \omega_{p-N-1} - (J+N+2)\omega_p + \omega_{n-q'} & \text{if } J = 0, \\ \omega_{p-\ell-N} - (J+N+1)\omega_p - \omega_{p+1} + \omega_{n-q'+1} & \text{if } J > 0. \end{cases}$$

By (7.6.2) we have

$$(\gamma(\mu+\rho) - \rho) - \lambda = \begin{cases} \omega_{p-N-1} - 2\omega_p + \omega_{p+N+1} & \text{if } J = 0, \\ \omega_{p-\ell-N} - \omega_p - \omega_{p+1} - \omega_{n-q'} + \omega_{n-q'+1} + \omega_{p+J+N+1} & \text{if } J > 0. \end{cases}$$

This agrees with expression given in the proposition for this case.

Finally, we consider the case $N > 0$ and $s_* < N+1$. As in the proof of (6.2), we must consider the cases $s_* = 1$ and $s_* > 1$ separately. Suppose first that $s_* = 1$. Then by (5.18.2) we have $a_{n-q'} > 1$ and by the proof of (6.2) μ has the form

$$\mu = \omega_{p-\ell} + \sum_{i=p}^{n-q'-2} a_{i+1}\omega_i + (a_{n-q'} - 2)\omega_{n-q'-1} + \omega_{n-q'}.$$

By (7.6.1), if $J = 0$ then

$$\gamma(\mu) = \omega_{p-\ell} + \Big(\sum_{i=p+1}^{n-q'} a_i - 2\Big)\omega_p - \Big(\sum_{i=p+2}^{n-q'} a_i - 2\Big)\omega_{p+1} + \sum_{i=p+2}^{n-q'-1} a_i\omega_i + (a_{n-q'} - 2)\omega_{n-q'} + \omega_{n-q'},$$

and if $J > 0$ then

$$\gamma(\mu) = \omega_{p-\ell} + \Big(\sum_{i=p+1}^{n-q'} a_i - 1\Big)\omega_p - \Big(\sum_{i=p+2}^{n-q'} a_i - 1\Big)\omega_{p+1} + \sum_{i=p+2}^{n-q'-1} a_i\omega_i + (a_{n-q'} - 2)\omega_{n-q'} + \omega_{n-q'+1}.$$

By (7.6.2) and the form of λ given in (7.5) we then obtain

$$(\gamma(\mu+\rho) - \rho) - \lambda = \begin{cases} \omega_{p-1} - 2\omega_p + \omega_{p+1} & \text{if } J = 0, \\ \omega_{p-\ell} - \omega_p - 2\omega_{n-q'} + \omega_{n-q'+1} + \omega_{n-q'+\ell-1} & \text{if } J > 0. \end{cases}$$

This proves the proposition in the case $s_* = 1$.

We now consider the final case $1 < s_* < N+1$. Then by (5.18.2) $a_{n-q'} = 1, a_{n-q'-1} = \cdots = a_{n-q'-s_*+2} = 0$ and $a_{n-q'-s_*+1} > 0$ so that by (7.5)

$$\lambda = -\Big(J + N + 1 + \sum_{i=p+2}^{n-q'-s_*+1} a_i\Big)\omega_{p+1} + \sum_{i=p+2}^{n-q'-s_*+1} a_i\omega_i + \omega_{n-q'}. \tag{7.10}$$

By the proof of (6.2) we have

$$\mu = \omega_{p+1-\ell-s_*} + \sum_{i=p}^{n-q'-s_*-1} a_{i+1}\omega_i + (a_{n-q'-s_*+1} - 1)\omega_{n-q'-s_*} + \omega_{n-q'}.$$

If $J = 0$, then by (7.6.1)

$$\gamma(\mu) = \omega_{p-s_*} - (N+2)\omega_p + \Big(\sum_{i=p+2}^{n-q'-s_*+1} a_i - 1\Big)\omega_{p+1} + \sum_{i=p+2}^{n-q'-s_*+1} a_i\omega_i - \omega_{n-q'-s_*+1} + \omega_{n-q'},$$

and if $J > 0$, then

$$\gamma(\mu) = \omega_{p+1-\ell-s_*} - (J+N+1)\omega_p - \Big(\sum_{i=p+2}^{n-q'-s_*+1} a_i\Big)\omega_{p+1} + \sum_{i=p+2}^{n-q'-s_*+1} a_i\omega_i - \omega_{n-q'-s_*+1} + \omega_{n-q'+1}.$$

By (7.6.2) and the form of λ given in (7.10) one has

$$(\sigma_{p+1}\cdots\sigma_{p+I}(\mu+\rho)-\rho)-\lambda = \begin{cases} \omega_{p-s_*} - 2\omega_p + \omega_{p+1} - \omega_{n-q'-s_*+1} + \omega_{n-q'} & \text{if } \ell = 1, \\ \omega_{p+1-\ell-s_*} - \omega_p - \omega_{n-q'-s_*+1} - \omega_{n-q'} + \omega_{n-q'+1} + \omega_{n-q'+\ell-1} & \text{if } \ell > 1. \end{cases}$$

This completes the proof of the proposition.

We now describe the decomposition of $S(\overline{\mathfrak{u}}\cap\mathfrak{p}_-)$ into M types using a well-known result of Schmid [17]. This result applies to the group $G_1 = U(p,q-1)$ and decomposes $S(\mathfrak{p}_1^-)$ where $\mathfrak{p}_1^- = \left\{\begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} : a \in \mathbb{C}^{(q-1)\times p}\right\}$ into $K_1 = U(p)\times U(q-1)$ types. Since $M \cong K_1$ and since there exists an M isomorphism $S(\overline{\mathfrak{u}}\cap\mathfrak{p}_-) \cong S(\mathfrak{p}_1^-)$, we obtain an M decomposition of $S(\overline{\mathfrak{u}}\cap\mathfrak{p}_-)$.

Set $r = \min\{p, q-1\}$ and define orthogonal roots by putting $\xi_i = -(\alpha_i + \cdots + \alpha_{n-i})$ for $1 \le i \le r$. Let $\mathcal{J}$ denote the set of integral multi-indices $\vec{j} = (j_1, \ldots, j_r)$ with $j_1 \ge j_2 \ge \cdots \ge j_r \ge 0$. Let $E_M(\vec{j})$ denote the irreducible finite-dimensional representation of M with lowest weight $\sum_{i=1}^r j_i\xi_i$.

(7.11) THEOREM [17]. *The space $S(\overline{\mathfrak{u}}\cap\mathfrak{p}_-)$ has the multiplicity free decomposition*

$$S(\overline{\mathfrak{u}}\cap\mathfrak{p}_-) = \bigoplus_{\vec{j}\in\mathcal{J}} E_M(\vec{j}).$$

By applying the element of $\mathcal{W}_{\mathfrak{m}}$, the Weyl group for $\mathfrak{m}$, of maximal length to the lowest weight, we find that $E_M(\vec{j})$ has highest weight

$$(7.12)\quad j_r\omega_{p-r} + (j_{r-1}-j_r)\omega_{p-r+1} + \cdots + (j_1-j_2)\omega_{p-1} + (-j_1)\omega_p + (-j_1)\omega_{p+1} + (j_1-j_2)\omega_{p+2} + \cdots + (j_{r-1}-j_r)\omega_{p+r} + j_r\omega_{p+r+1}.$$

In the case $r = p$ (resp. $r = q-1$) it is understood that the corresponding undefined term $j_p\omega_0$ (resp. $j_{q-1}\omega_n$) is deleted from (7.12). We let $F_M(\vec{j})$ denote the irreducible M space with highest weight given by (7.12).

Every irreducible representation of $U(p)\times U(q-1)$ is equivalent to a tensor product of an irreducible representation of $U(p)$ with an irreducible representation of $U(q-1)$ (cf. p.45, [15]). Since $M \cong U(p)\times U(q-1)$, we denote this fact by writing

$$(7.13)\qquad F_M\Big(\sum_{i=1}^{n-1} b_i\omega_i\Big) = F_1\Big(\sum_{i=1}^{p} b_i\omega_i\Big)\otimes F_2\Big(\sum_{i=p+1}^{n-1} b_i\omega_i\Big),$$

where $F_1(\sum_{i=1}^{p} b_i\omega_i)$ denotes the irreducible space for the block-diagonal subgroup of M isomorphic to $U(p)$ with highest weight $\sum_{i=1}^{p} b_i\omega_i$ and $F_2(\sum_{i=p+1}^{n-1} b_i\omega_i)$ denotes the irreducible space for the block-diagonal subgroup of M isomorphic to $U(q-1)$ with highest weight $\sum_{i=p+1}^{n-1} b_i\omega_i$. We refer to $F_1(\sum_{i=1}^{p} b_i\omega_i)$ as the $U(p)$ part and $F_2(\sum_{i=p+1}^{n-1} b_i\omega_i)$ as the $U(q-1)$ part of the M type in (7.13). Here the coefficients b_i are integers for $1 \le i \le n-1$ and are nonnegative integers for $i \ne p, p+1$. Irreducible representations of $U(p)$ are typically parametrized by the signature $(k_1, \dots, k_p)$ where $k_j \in \mathbb{Z}, 1 \le j \le p$, satisfy $k_1 \ge k_2 \ge \cdots \ge k_p$. The signature of the $U(p)$ part in (7.13) is given by $k_j = \sum_{i=j}^{p} b_i$ and the signature $(k_1, \cdots, k_{q-1})$ of the $U(q-1)$ part in (7.13) is given by $k_j = -\sum_{i=1}^{j} b_{p+i}$.

Define elements Z_p and Z_{q-1} in $Z(\mathfrak{m})$ by $Z_p = \sum_{i=1}^{p} ih_i$ and $Z_{q-1} = -\sum_{i=1}^{q-1}(q-i)h_{p+i}$. It follows that Z_p acts on the $U(p)$ part of (7.13) by the scalar $\sum_{i=1}^{p} ib_i$ and Z_{q-1} acts on the $U(q-1)$ part of (7.13) by the scalar $-\sum_{i=1}^{q-1}(q-i)b_{p+i}$.

(7.14) PROPOSITION. *Let λ in (5.7) be a reduction point. Assume $p' = p+1$ and $s_* \ge 1$. Let μ denote the weight of the CDO pair of (6.2) and let $I = J+N$. Then for all $m \ge 0$ one has*

$$\operatorname{Hom}_M(F_M(\sigma_{p+1}\cdots\sigma_{p+I}(\mu+\rho)-\rho), S^m(\bar{\mathfrak{u}}\cap\mathfrak{p}_-)\otimes F_L(\lambda)) = 0.$$

PROOF. Fix $m \ge 0$ and write $S^m(\bar{\mathfrak{u}}\cap\mathfrak{p}_-) = \oplus_i V^i$ where V^i is an irreducible representation space for M. As in (7.13) write $V^i = V_1^i \otimes V_2^i$ where V_1^i (resp. V_2^i) is an irreducible representation space of $U(p)$ (resp. $U(q-1)$). Since $p' = p+1$, we have $F_L(\lambda) = F_1(0)\otimes F_2(\lambda)$ so that

$$S^m(\bar{\mathfrak{u}}\cap\mathfrak{p}_-)\otimes F_L(\lambda) = \bigoplus_i V_1^i \otimes (V_2^i \otimes F_2(\lambda)).$$

Consequently, each irreducible component in $S^m(\bar{\mathfrak{u}}\cap\mathfrak{p}_-)\otimes F_L(\lambda)$ must have a $U(p)$ part which matches the $U(p)$ part of an irreducible component of $S^m(\bar{\mathfrak{u}}\cap\mathfrak{p}_-)$. Put $\chi = \sigma_{p+1}\cdots\sigma_{p+I}(\mu+\rho)-\rho$ and

$$\beta(\ell, s_*) = \begin{cases} \omega_{p+1} - \omega_{n-q'-s_*+1} + \omega_{n-q'} & \text{if } \ell = 1, \\ -\omega_{n-q'-s_*+1} - \omega_{n-q'} + \omega_{n-q'+1} + \omega_{n-q'+\ell-1} & \text{if } \ell > 1 \end{cases} \tag{7.15}$$

By (7.7), we have

$$F_M(\chi) = \begin{cases} F_1(\omega_{p-s_*} - 2\omega_p)\otimes F_2(\lambda+\beta(\ell,s_*)) & \text{if } \ell = 1, \\ F_1(\omega_{p+1-\ell-s_*} - \omega_p)\otimes F_2(\lambda+\beta(\ell,s_*)) & \text{if } \ell > 1. \end{cases} \tag{7.16}$$

We first consider the case $\ell = 1$. By (7.16) the $U(p)$ part of $F_M(\chi)$ has highest weight $\omega_{p-s_*} - 2\omega_p$ if $s_* < p$ and highest weight $-2\omega_p$ if $s_* = p$. If $s_* < p$ then by the form of the weight in (7.12) there is no M type in $S(\bar{\mathfrak{u}}\cap\mathfrak{p}_-)$ with $U(p)$ part of the form $\omega_{p-s_*} - 2\omega_p$. By the above discussion, $F_M(\chi)$ is not an M type in $S^m(\bar{\mathfrak{u}}\cap\mathfrak{p}_-)\otimes F_L(\lambda)$ for any $m \ge 0$. If $s_* = p$, then the only M type in $S(\bar{\mathfrak{u}}\cap\mathfrak{p}_-)$ with $U(p)$ part having highest weight $-2\omega_p$ occurs when $p = r$ and $\vec{j} = (2, \dots, 2)$. Write $F_M(\vec{j}) = F_1(\vec{j})\otimes F_2(\vec{j})$ and note that the

central element Z_{q-1} acts by $2p$ on $F_2(\vec{j})$. Recall that the highest weights of the irreducible components of $F_2(\vec{j}) \otimes F_2(\lambda)$ are of the form $\lambda + Q$ where Q is a weight of $F_2(\vec{j})$. Thus, if $F_M(\chi)$ is an M type in $F_M(\vec{j}) \otimes F_L(\lambda)$, then $\beta(\ell, s_*)$ must be a weight of $F_2(\vec{j})$. But by (7.15) we find that Z_{q-1} acts by $s_* - q = p - q$ on a weight vector with weight $\beta(\ell, s_*)$. Since $2p \neq p - q$, we conclude that $\beta(\ell, s_*)$ is not a weight of $F_2(\vec{j})$ so that $F_M(\chi)$ is not an M type in $F_M(\vec{j}) \otimes F_L(\lambda)$. This completes the proof of the $\ell = 1$ case.

Assume $\ell > 1$. The highest weight of the $U(p)$ part of $F_M(\chi)$ is $\omega_{p+1-\ell-s_*} - \omega_p$ if $\ell + s_* < p+1$ and is $-\omega_p$ if $\ell + s_* = p+1$. Consider first the case $\ell + s_* = p+1$. By (7.12) the only M type in $S(\bar{\mathfrak{u}} \cap \mathfrak{p}_-)$ with $U(p)$ part having highest weight $-\omega_p$ occurs when $p = r$ and $\vec{j} = (1, \ldots, 1)$. Write $F_M(\vec{j}) = F_1(\vec{j}) \otimes F_2(\vec{j})$. Just as in the argument for the $\ell = 1$ case, we know that if $F_M(\chi)$ is an M type in $F_M(\vec{j}) \otimes F_L(\lambda)$, then $\beta(\ell, s_*)$ must be a weight of $F_2(\vec{j})$. By (7.12), there are two possibilities for the highest weight of $F_2(\vec{j})$. If $2p + 1 = n$, then $F_2(\vec{j})$ has highest weight $-\omega_{p+1}$ and if $2p + 1 < n$, then $F_2(\vec{j})$ has highest weight $-\omega_{p+1} + \omega_{2p+1}$. If $2p+1 = n$, then $F_2(\vec{j})$ is one-dimensional with weight $-\omega_{p+1}$. Since $\beta(\ell, s_*) \neq -\omega_{p+1}$, we conclude that $F_M(\chi)$ is not an M type in $F_M(\vec{j}) \otimes F_L(\lambda)$. If $2p + 1 < n$, then $F_2(\vec{j})$ is equivalent to the fundamental representation $F(e_1 + \cdots + e_p)$ of $U(q-1)$ with highest weight $e_1 + \cdots + e_p$. But the weights of $F(e_1 + \cdots + e_p)$ are sums $e_{i_1} + \cdots + e_{i_p}$ where $1 \leq i_1 < \cdots < i_p \leq q-1$. Let $\beta'(\ell, s_*)$ denote the weight for $U(q-1)$ corresponding to $\beta(\ell, s_*)$ in (7.15). Then for $\ell > 1$ we have

$$\beta'(\ell, s_*) = e_{q-q'-s_*+1} + \cdots + e_{q-q'+\ell-2} + e_{q-q'}. \tag{7.17}$$

Since $\ell > 1$, we have $q - q' - s_* + 1 \leq q - q' \leq q - q' + \ell - 2$ so that (7.17) does not give a weight of $F(e_1 + \cdots + e_p)$ since $e_{q-q'}$ appears twice. We conclude that $F_M(\chi)$ is not an M type in $S(\bar{\mathfrak{u}} \cap \mathfrak{p}_-) \otimes F_L(\lambda)$ in the case $2p + 1 < n$.

Finally, we consider the case $\ell + s_* < p + 1$. As above, we check to see whether $\beta(\ell, s_*)$ is a weight of $F_2(-\omega_{p+1} + \omega_{p+\ell+s_*})$. But $F_2(-\omega_{p+1} + \omega_{p+\ell+s_*})$ is equivalent to the fundamental representation $F(e_1 + \cdots + e_{\ell+s_*-1})$ of $U(q-1)$. Again by (7.17), we see that $\beta'(\ell, s_*)$ cannot be a weight in $F(e_1 + \cdots + e_{\ell+s_*-1})$. We conclude that $F_M(\chi)$ is not an M type in $S(\bar{\mathfrak{u}} \cap \mathfrak{p}_-) \otimes F_L(\lambda)$ in the case $\ell + s_* < p + 1$. This completes the proof of the proposition.

(7.18) THEOREM. *Let λ in (5.7) be a reduction point. Assume $p' = p + 1$ and $s_* \geq 1$. Set $\sigma = \sigma_{n-q'-1} \cdots \sigma_{p-\ell+2}$. Then*

$$\Gamma^{\ell(\sigma)}\left(\mathrm{pro}_{\mathfrak{q},M}^{\mathfrak{g},M}(F_L(\lambda))\right) = \mathrm{Ker}(D_{\tau,\mu})$$

where $D_{\tau,\mu}$ is a nonzero CDO whose parameters are given by the CDO pairs (τ, μ) in (6.2). Consequently, $\Gamma^{\ell(\sigma)}\left(\mathrm{pro}_{\mathfrak{q},M}^{\mathfrak{g},M}(F_L(\lambda))\right)$ is an irreducible unitarizable highest weight $(\mathfrak{g}, K)$ module with highest weight $\tau = \sigma(\lambda + \rho) - \rho$.

PROOF. Let (τ, μ) be the CDO pair in (6.2) and let Z denote the irreducible representation space corresponding to μ. By (7.14) the E_2 terms in (7.1) converging to $\mathrm{Ext}^I_{\mathfrak{k},M}(Z, \mathrm{pro}^{\mathfrak{g},M}_{\mathfrak{q},M}(F_L(\lambda)))$ are zero so this Ext group is zero. But the dimension of this Ext group is the multiplicity of Z in $\Gamma^{\ell(\sigma)}\left(\mathrm{pro}^{\mathfrak{g},M}_{\mathfrak{q},M}(F_L(\lambda))\right)$ by (6.2.15) [18]. Since Z occurs as a K type in $\mathrm{pro}^{\mathfrak{g},K}_{\overline{\mathfrak{q}(\omega_p)},K}(F_K(\tau))$ we know that $\Gamma^{\ell(\sigma)}\left(\mathrm{pro}^{\mathfrak{g},M}_{\mathfrak{q},M}(F_L(\lambda))\right)$ is proper in $\mathrm{pro}^{\mathfrak{g},K}_{\overline{\mathfrak{q}(\omega_p)},K}(F_K(\tau))$. Note that the assumption $p' = p+1$ implies that $I < q-1$ if and only if $\ell \leq q'$. Since λ is a reduction point, we have $\ell \leq \min\{p', q'\}$. Consequently, (6.5) is applicable and we conclude that $\Gamma^{\ell(\sigma)}\left(\mathrm{pro}^{\mathfrak{g},M}_{\mathfrak{q},M}(F_L(\lambda))\right)$ is irreducible.

8. A second parabolic

In this section we state the results obtained by replacing the parabolic $\mathfrak{q}(\omega_{p+1})$ used in the previous sections with the parabolic $\mathfrak{q} = \mathfrak{q}(\omega_{p-1})$. The groups L and M and reduction points λ are replaced with their counterparts which we now describe.

Assume $p > 1$ and set $L = S(U(p-1) \times U(q+1))$. Then L is the maximal compact subgroup of $SU(p-1, q+1)$ and $M = L \cap K \cong U(p-1) \times U(q)$. The reduction points λ for $SU(p-1, q+1)$ then take the form

$$\lambda = \sum_{i=p'}^{p-2} a_i\omega_i - \Big(n - p' - q' + \ell - 1 + \sum_{i \neq p-1} a_i\Big)\omega_{p-1} + \sum_{i=p}^{n-q'} a_i\omega_i \tag{8.1}$$

where $1 \leq p' \leq p-1$, $1 \leq q' \leq q+1$ and $1 \leq \ell \leq \min\{p', q'\}$. It is understood that $p' = p-1$ iff $a_1 = \cdots = a_{p-2} = 0$ and $q' = q+1$ iff $a_p = \cdots = a_{n-1} = 0$.

For λ as in (8.1) we define weights $\lambda_j, 0 \leq j \leq q$, by $\lambda_j = \sigma_p \cdots \sigma_{p+j-1}(\lambda + \rho) - \rho$, for $1 \leq j \leq q$, and $\lambda_0 = \lambda$. Define integers J, N and $s_* \in \{0, \ldots q-J\}$ by $J = q - q' + \ell, N = p - p' - 1$ and

$$s_* = \begin{cases} \min \Sigma & \text{if } \Sigma \neq \varnothing, \\ 0 & \text{if } \Sigma = \varnothing, \end{cases} \tag{8.2}$$

where $\Sigma = \{s > 0 : \lambda_{J+s} \text{ is } \mathfrak{k} \text{ nonsingular}\}$. The following result is analogous to (5.18).

(8.3) PROPOSITION. *Let λ in (8.1) be a reduction point and suppose $s_* \geq 1$.*

(1) *If $N = 0$, then $s_* = 1$.*

(2) *Assume $N > 0$. Then $s_* = 1$ if and only if $a_{p'} > 1$. If $a_{p'} = 1$, then $s_* \leq N+1$ and*

$$s_* = \begin{cases} \min(Q) + 1 & \text{if } Q \neq \varnothing, \\ N+1 & \text{if } Q = \varnothing, \end{cases}$$

where $Q = \{i : 1 \leq i \leq N-1 \text{ and } a_{p'+i} > 0\}$.

(8.4) PROPOSITION. *Let λ in (8.1) be a reduction point and assume $s_* \geq 1$. Put*

$$\tau = \begin{cases} \lambda_J & \text{if } N = 0, \\ \sigma_{p-N} \cdots \sigma_{p-1}(\lambda_J + \rho_k) - \rho_k & \text{if } N > 0, \end{cases}$$

$$\mu = \begin{cases} \lambda_{J+1} & \text{if } N = 0, \\ \lambda_{J+N+1} & \text{if } N > 0 \text{ and } s_* = N+1, \\ \sigma_{p-N+s_*-1} \cdots \sigma_{p-1}(\lambda_{J+s_*} + \rho_k) - \rho_k & \text{if } N > 0 \text{ and } s_* < N+1. \end{cases}$$

Then (τ, μ) is a CDO pair for $SU(p,q)$ with cone data $(p' + s_*, q' - \ell, s_*)$.

(8.5) THEOREM. *Let λ in (8.1) be a reduction point. Assume $s_* \geq 1$ and $J + N < p - 1$.*

(1) *Define $\sigma \in \mathcal{W}$ by $\sigma = \sigma_{p-N} \cdots \sigma_{p+J-1}$. With the differential notation of section five we have*

$$\operatorname{Ker}(d^{J,N}_{s_*}) = \Gamma^{\ell(\sigma)}\Big(\operatorname{pro}^{\mathfrak{g},M}_{\mathfrak{q},M}(F_L(\lambda))\Big).$$

(2) *The differential $d^{J,N}_{s_*}$ is a multiple of the CDO $D_{\tau,\mu}$ whose parameters are given by the CDO pairs (τ, μ) in (8.4).*

(3) *If $q' = q + 1$, then $\Gamma^{\ell(\sigma)}\Big(\operatorname{pro}^{\mathfrak{g},M}_{\mathfrak{q},M}(F_L(\lambda))\Big)$ is an irreducible, unitarizable highest weight $(\mathfrak{g}, K)$ module with highest weight $\tau = \sigma(\lambda + \rho) - \rho$.*

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REFERENCES

[1] R. J. Baston and M. G. Eastwood, *The Penrose Transform: Its Interaction with Representation Theory*, Clarendon Press, Oxford, 1989.

[2] B. Boe and D. Collingwood, *A multiplicity one theorem for holomorphically induced representations*, Math. Z. **192** (1986), 252-282.

[3] B. Boe, T. J. Enright and B. Shelton, *Determination of the interwining operators for holomorphically induced representations of Hermitian symmetric pairs*, Pac. J. Math. **131** (1988), 39-50.

[4] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, Princeton, N.J., 1956.

[5] M. G. Davidson, T. J. Enright, and R. J. Stanke, *Differential operators and highest weight representations*, Memoirs Amer. Math. Soc. **455** (1991).

[6] M. G. Davidson, T. J. Enright and R. J. Stanke, *Covariant differential operators*, Math. Ann. **288** (1990), 731-739.

[7] M. G. Eastwood, R. Penrose, and R. O. Wells, Jr., *Cohomology and massless fields*, Commum. Math. Phys. **78** (1981), 305-351.

[8] T. J. Enright, R. Howe, and N. R. Wallach, *A classification of unitary highest weight modules*, In: *Representation Theory of Reductive Groups*, Birkhäuser, Boston, MA, 1983.

[9] T. J. Enright, R. Parthasarathy, N. R. Wallach and J. A. Wolf, *Unitary derived functor modules with small spectrum*, Acta Math. **154** (1985), 105-136.

[10] T. J. Enright and N. R. Wallach, *Notes on homological algebra and representations of Lie algebras*, Duke Math. J. **47** (1980), 1-14.
[11] A. W. Knapp, *Representation Theory of Semisimple Groups: An Overview Based on Examples*, Princeton University Press, Princeton N.J., 1986.
[12] A. W. Knapp, *Lie Groups, Lie Algebras, and Cohomology*, Math. Notes 34, Princeton University Press, Princeton N.J., 1988.
[13] J. Lepowsky, *A generalization of the Bernstein-Gelfand-Gelfand resolution*, J. Algebra **49** (1977), 496-511.
[14] P. Moseneder Frajria, *Derived functors of unitary highest weight modules at reduction points*, Trans. Amer. Math. Soc. **327** (1991), 703-738.
[15] M. A. Naimark and A. I. Stern, *Theory of Group Representations*, Springer-Verlag, New York, 1982.
[16] J. J. Rotman, *An Introduction to Homological Algebra*, Academic Press, Orlando, FL, 1979.
[17] W. Schmid, *Die randwerte holomorpher funktionen auf hermitesch symmetrischen räumen*, Invent. Math. **9** (1969/70), 61-80.
[18] D. A. Vogan, *Representations of Real Reductive Groups*, Progress in Math. 15, Birkhäuser, Boston, MA, 1981.
[19] N. R. Wallach, *Real Reductive Groups, I*, Academic Press, Boston, MA, 1988.
[20] N. R. Wallach, *Transfer of unitary representations between real forms*, In: *Representation Theory and Analysis on Homogeneous Spaces, Contemporary Mathematics*, **177** (1994), S. G. Gindikin *et al.* eds.
[21] D. P. Zelobenko, *Compact Lie Groups and Their Representations*, Translations Math. Monographs, vol. 40, Amer. Math. Soc., Providence, RI, 1973.

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Dual Representations and Invariant Theory

TUONG TON-THAT

Dedicated to Ray A. Kunze

ABSTRACT. Let R (resp. R') be a completely reducible representation of a group G (resp. G') on a Hilbert space $\mathcal{H}$ in such a way that the two actions commute. Then R and R' are said to be *dual representations* if the spectral decomposition of one determines completely that of the other and *vice-versa*. The Schur-Weyl duality between symmetric and general linear groups, the complementary pairs of groups introduced by Moshinsky and Quesne, and the reductive dual pairs classified by Howe are typical examples of dual representations. Examples of very unusual dual representations which include semi-direct products of groups and infinite-dimensional Lie algebras are exhibited.

Given a representation R of a group G on a Hilbert space $\mathcal{H}$ we show, using the theory of polynomial invariants under R, how to find its dual representation R' of a group G'. Conversely, given a pair of dual representations we use the invariants of one member to find the invariants of the other.

Several theorems for dual representations and several theorems in invariant theory including some on generalized Casmimir invariants are proved.

1. Introduction

In the preface of his book [We] "The classical groups, their invariants and representations" H. Weyl wrote: "... The notion of an algebraic invariant of an abstract group γ cannot be formulated until we have before us the concept of a representation of γ. The problem of finding all representations of γ must therefore logically precede that of finding all algebraic invariants of γ ..." Since it's publication many treatises on invariant theory and group representations were written by mathematicians and physicists. One of the most striking examples of this intimate relation between these two concepts is illustrated by the following

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theorem of Racah [Ra] and Chevalley [Ch]: *For every semisimple Lie algebra* $\mathfrak{g}$ *of rank* n *there exists a set of* n *algebraically independent generators of the Casimir invariants whose eigenvalues characterize the finite-dimensional irreducible representations of* $\mathfrak{g}$. *Moreover, the product of the degrees of all these generators is equal to the order of "the" Weyl group of* $\mathfrak{g}$.

The central theme of this article is in a sense to reverse Weyl's procedure. This can be somewhat imprecisely stated as follows: Let R be a reducible representation of a Lie group G on a Hilbert space $\mathcal{H}$ which contains a subspace of polynomial functions P. Assume R is "rich enough" which, in the optimal case, means that R would contain all equivalence classes of irreducible representations of G. Find a minimal set of generators of R-invariants in P, and from this set derive a representation R' on $\mathcal{H}$, of a Lie group G', which is "dual" to R. In general, R' is also reducible and rich enough. Finally, use the knowledge of the Casimir invariants of R to find the Casimir invariants of R'.

2. Dual Representations

The Schur-Weyl Duality Theorem (cf. [Sc] and [We]) is the first theorem, as far as we know, that classifies, up to isomorphism, all irreducible representations of the symmetric group of degree n in terms of all irreducible rational representations of the general linear group of order n. This idea of characterizing irreducible representations of one group by that of another group was taken up by the physicists M. Moshinsky and C. Quesne who introduced the notion of *complementary pairs* in [MQ]. This was simultaneously generalized by K. Gross and R. Kunze in a series of papers starting with [GK1] and [GK2]. The idea was further developed in [Sa], [KV], [Ge], [EHW],.... Finally, it was R. Howe who gave the general definition and classification of *reductive dual pairs* in [Ho1] and [Ho2] (this paper [Ho1] was written in 1976 and had been widely circulated as a preprint). We felt, however, that this class of reductive dual pairs does not cover the pairs that occur in the Schur-Weyl Duality Theorem, the Peter-Weyl theorem, etc..., and gave a general definition of dual representation in [KT1] which is recalled here.

DEFINITION 2.1. Let G and G' be two groups. Let R (resp. R') be a representation of G (resp. G') on a vector space $\mathcal{H}$ such that the two actions commute and both representations are completely reducible. Then R *and* R' are called *dual representations* if the spectral decomposition of R determines that of R' completely and vice-versa. The analogous definition of dual representations for Lie algebras obviously holds.

Aside from the examples given in [KT1], it is worthwhile to mention here two more. The first one, given by K. Nishiyama in [Ni], establishes the correspondence between irreducible unitary representations of the orthosymplectic Lie super algebra $\mathfrak{osp}(2/1;\mathbb{R})$ and certain finite-dimensional irreducible representations of $\mathfrak{so}(n)$. The second one was privately communicated to us by G. Savin (March 1993) and we shall describe it below:

Let $\mathfrak{g} = \mathfrak{g}' = \mathfrak{sl}(2,\mathbb{C})$, let $\mathbb{C}(x,y,z)$ be the space of rational functions in (x,y,z), and let $\mathcal{H} \equiv \mathbb{C}(x,y,z)_K$ denote the space of K- finite vectors in $\mathbb{C}(x,y,z)$ $(K = SU(2))$. Let $\{E,F,H\}$ denote the standard basis of $\mathfrak{g}$ with commutation relations

$$[H,E] = 2E\ , \qquad [H,F] = -2F\ , \qquad [E,F] = H\ .$$

Define the representations R and R' on $\mathcal{H}$ by

$$\begin{cases} R(E) = \frac{\partial}{\partial y}, & R(F) = nxy\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y} + (n-1)y - \frac{1}{nx^2}(\frac{\partial}{\partial y})^n\frac{\partial}{\partial z} \\ R'(E) = \frac{\partial}{\partial z}, & R'(F) = xz\frac{\partial}{\partial x} - z^2\frac{\partial}{\partial z} - \frac{1}{n^3x^2}(\frac{\partial}{\partial y})^n\ . \end{cases}$$

where n is a positive integer. Then the $R \times R'$ module $\mathcal{H}$ is decomposed into irreducible submodules as

$$\mathcal{H} = \sum_{k=1}^{\infty} \oplus \mathcal{I}^{(k)}\ ,$$

where $\mathcal{I}^{(k)}$, the (k)-*isotypic component*, is isomorphic to $V_k \otimes V_{nk}$, where V_i denotes the irreducible i-dimensional module of $\mathfrak{sl}(2,\mathbb{C})$. The corresponding highest weight vector of $\mathcal{I}^{(k)}$ is x^{k-1}.

The next class of examples, which shall be discussed from Section 4 onward, constitutes the central object of this article. It involves the theory of polynomial invariants, of which we shall discuss in the next section, the parts pertaining to our work.

3. Invariant Theory

Let V be a finite-dimensional vector over $\mathbb{R}$ (or $\mathbb{C}$), let $P(V) \equiv S(V^*)$ denote the algebra of polynomial functions on V. Let ρ be a representation of a connected Lie group G on V. Then

DEFINITION 3.1. A polynomial function $p \in P(V)$ is said to be ρ-*invariant* if $p(\rho(g)v) = p(v)$ for all $v \in V$ and $g \in G$. We have the following important example of polynomial invariants:

Let $\mathfrak{g}$ denote the Lie algebra of G, and let $\mathfrak{g}^*$ denote its dual. Take V to be $\mathfrak{g}^*$ and ρ to be the coadjoint representation of G on V. Clearly $V^* = \mathfrak{g}$ so that $\mathfrak{g}$ generates $P(V)$. It follows that for every $X \in \mathfrak{g}$ there exists a unique derivation of $P(V)$ which, when restricted to $\mathfrak{g}$, reduces to $ad(X)$, and that the invariants of the coadjoint representation of G are the elements of $P(V)$ which are annihilated by these derivations. Let $\mathcal{U} \equiv \mathcal{U}(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. Let $\lambda : S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ denote the *symmetrizer* (or *canonical map*), then λ is a $\mathfrak{g}$-module isomorphism. Let $\mathcal{Z} \equiv \mathcal{Z}(\mathcal{U})$ denote the center of $\mathcal{U}$; i.e., $\mathcal{Z} = \{u \in \mathcal{U} : ad(X)u = 0, \forall X \in \mathfrak{g}\}$. Let I denote the subalgebra of all coadjoint-invariants in $P(V) \equiv S(\mathfrak{g})$ then $\lambda(I) = \mathcal{Z}$ (cf. [Di], Chap. 4 or [Go]). Elements of $\mathcal{Z}$ are called by the physicists *Casimir invariants* (the quadratic Casimir invariant is the classical Casimir operator or Casimir element). The importance of these Casimir invariants is due in part to the fundamental theorem of Chevalley and Racah stated in the introduction.

REMARK 3.2. If G is a real Lie group we let $\mathfrak{g}^{\mathbb{C}}$ denote the complexification of $\mathfrak{g}$ and regard $\mathfrak{g}$ as a subset of $\mathfrak{g}^{\mathbb{C}}$. Let $\mathcal{U}^{\mathbb{C}}$ denote the universal enveloping algebra of $\mathfrak{g}^{\mathbb{C}}$ and let $\mathcal{U}$ be the subalgebra (over $\mathbb{R}$) generated by 1 and $\mathfrak{g}$. Then we may identify $\mathcal{U}$ with the universal enveloping algebra of $\mathfrak{g}$ and choose a set of generators of the Casimir invariants in $\mathcal{U}$ so that it constitutes also a set of generators of the Casimir invariants in $\mathcal{U}^{\mathbb{C}}$. This follows from the well-known fact that a polynomial function in several complex variables is uniquely determined by its restriction to the corresponding real variables.

4. Some General Theorems on Invariant Theory and the Oscillator Dual Representations

Henceforth we shall specialize to the following category of dual representations:

Let $\mathbb{C}^{n\times N}$ denote the vector space of all $n \times N$ complex matrices. Define a Gaussian measure μ on $\mathbb{C}^{n\times N}$ by

$$d\mu(Z) = \pi^{-nN} \exp[-Tr(ZZ^*)]dZ \ , \quad Z \in \mathbb{C}^{n\times N} \tag{4.1}$$

where in Eq. (4.1) dZ denotes the Lebesgue measure on $\mathbb{C}^{n\times N}$. Let $\mathcal{F}(\mathbb{C}^{n\times N})$ denote the Fock space of entire functions on $\mathbb{C}^{n\times N}$ which are square integrable with respect to $d\mu$. Endowed with the inner product

$$(f|g) = \int_{\mathbb{C}^{n\times N}} f(Z)\overline{g(Z)}d\mu(Z); f, g \in \mathcal{F}(\mathbb{C}^{n\times N}) \ , \tag{4.2}$$

$\mathcal{F}(\mathbb{C}^{n\times N})$ has a Hilbert space structure. It can be easily verified that the inner product defined by Eq. (4.2) coincides with the following inner product

$$\langle f, g\rangle = f(D)\overline{g(\bar{Z})}\Big|_{Z=0} \ , \tag{4.3}$$

where $f(D)$ denotes the formal power series obtained by replacing $Z_{\alpha j}$ by the partial derivative $\partial/\partial Z_{\alpha j} (1 \leq \alpha \leq n \ , \ 1 \leq j \leq N)$. Note that the subspace $P(\mathbb{C}^{n\times N})$ of all polynomial functions on $\mathbb{C}^{n\times N}$ is dense in $\mathcal{F}(\mathbb{C}^{n\times N})$. In the theorems proved in this article the notion of dual representations takes a more precise form. Let R (resp. R') be an unitary representation of G (resp. G') on $\mathcal{F}(\mathbb{C}^{n\times N})$. As a representation space of the joint action $R \otimes R'$ the Hilbert space $\mathcal{F}(\mathbb{C}^{n\times N})$ is decomposed into an orthogonal direct sum

$$\mathcal{F}(\mathbb{C}^{n\times N}) = \sum_{(\lambda)} \oplus \mathcal{I}^{(\lambda)}(\mathbb{C}^{n\times N}) \tag{4.4}$$

where in Eq. (4.4) the label (λ) characterizes both an equivalence class of an irreducible representation λ_G of G and an equivalence class of an irreducible representation $\lambda'_{G'}$ of G', and $\mathcal{I}^{(\lambda)} \equiv \mathcal{I}^{(\lambda)}(\mathbb{C}^{n\times N})$ denotes the (λ)-*isotypic component*, i.e., the direct sum of all irreducible subrepresentations of R (resp. R') that belong to the equivalence class λ_G (resp. $\lambda'_{G'}$). Moreover the restriction of

$R \otimes R'$ to $\mathcal{I}^{(\lambda)}$ is irreducible; i.e., $\mathcal{I}^{(\lambda_G)} \approx V^{(\lambda_G)} \otimes W^{(\lambda'_{G'})}$, where $V^{(\lambda_G)}$ (resp. $W^{(\lambda'_{G'})}$) is an irreducible G-module of class (λ) (resp. G'-module of class (λ')).

In [Ho3] Howe proved several general reciprocity theorems for see-saw reductive dual pairs. However these theorems do not cover the class of dual representations that we will discuss later. To see this let us consider the following example (for more examples see [KLT]).

Set $G = U(N)$ and $G' = U(n)$ and let $R' \otimes R$ denote the joint action of $G' \times G$ on $\mathcal{F}(\mathbb{C}^{n \times N})$ by

$$[(R' \otimes R)(g', g)f](Z) = f(g'^{-1} Z g)$$

for all $(g', g) \in G' \times G$ and $f \in \mathcal{F}(\mathbb{C}^{n \times N})$. Then the representations R' and R are dual and we have the following decomposition

$$\mathcal{F}(\mathbb{C}^{n \times N}) = \sum_{(\lambda)} \oplus \mathcal{I}^{(\lambda)} \tag{4.5}$$

where the label (λ) denotes both a signature of an irreducible representation of G' and G of the form $(m_1, m_2, \dots)$ with $m_1 \geq m_2 \dots \geq 0$. We identify $H := U(N-k)$, $1 \leq k \leq N$ with the subgroup of G consisting of all matrices of the form $\begin{pmatrix} \gamma & \vdots & 0 \\ \dots & \dots & \dots \\ 0 & \vdots & I_k \end{pmatrix}$, $\gamma \in U(N-k)$. Let R_H denote the restriction of R to H, then it was proved in [KLT] that the dual representation of R_H is the representation $R'_{H'} := R' \ltimes_\tau \pi_1$ of $H' := G' \ltimes_\tau H_{n,k}$ where $H_{n,k}$ is a direct product of k copies of a Heisenberg group acting on the Fock space $\mathcal{F}(\mathbb{C}^{n \times k})$ (For the Fock realization of irreducible representations of Heisenberg groups see, e.g., [VK] Chap. 12, also this irreducible unitary representation π_1 will be discussed again in later sections of this present article). Clearly the pair (H', H) is not a reductive dual pair classified by Howe, however we are interested in having a reciprocity theorem for the pairs (H', H) and (G', G).

THEOREM 4.1. *We preserve the notations and assumptions of this section. Let R_G and $R_{G'}$ be dual representations, let H be a subgroup of G and let R_H denote the restriction of R_G to its subgroup H. Assume that the representation R_H is also completely reducible and that the dual representation of R_H is an unitary representation $R'_{H'}$ of a group $H' \supset G'$ on $\mathcal{F}(\mathbb{C}^{n \times N})$. Then we have the following decompositions of $\mathcal{F}(\mathbb{C}^{n \times N})$ into isotypic components*

$$\mathcal{F}(\mathbb{C}^{n \times N}) = \sum_{(\lambda)} \oplus \mathcal{I}^{(\lambda)} = \sum_{(\mu)} \oplus \mathcal{I}^{(\mu)}\ , \tag{4.6}$$

where (λ) is a common irreducible label for the pair (G', G) and (μ) is a common irreducible label for the pair (H', H).

If λ_G (resp. $\lambda'_{G'}$) denotes an irreducible representation of the class (λ) and μ_H (resp. $\mu'_{H'}$) denotes an irreducible representation of the class (μ) then the multiplicity $dim[Hom_H(\mu_H : \lambda|_H)]$ of the irreducible representation μ_H in the restriction to H of the representation λ_G is equal to the multiplicity $\dim[Hom_{G'}(\lambda'_{G'} :$

$\mu'|_{G'})]$ *of the irreducible representation* $\lambda'_{G'}$ *in the restriction to* G' *of the representation* $\mu'_{H'}$.

PROOF. In Eq. (4.6) if the isotypic component $\mathcal{I}^{(\lambda)}$ is viewed, under the restriction of the action of G to H, as a H-submodule then obviously $\mathcal{I}^{(\lambda)} \cap \mathcal{I}^{(\mu)}$ is a H-submodule of the H-module $\mathcal{F}(\mathbb{C}^{n\times N})$. By the assumption of the complete reducibility of R_H we have the following orthogonal direct sum decomposition

$$\mathcal{I}^{(\lambda)} \cap \mathcal{I}^{(\mu)} = \sum_{a,b} \oplus V_{a,b}^{(\mu_H)} , \tag{4.7}$$

where $V_{a,b}^{(\mu_H)}$ is an irreducible H-module of class (μ), the index "a" ranges over a set of cardinality equal to $dim[Hom_H(\mu_H : \lambda|_H)]$ and the index b ranges over a set of cardinality equal to $mult(\lambda_G; R_G)$, the multiplicity of λ_G in R_G. Similarly if we consider $\mathcal{I}^{(\lambda)} \cap \mathcal{I}^{(\mu)}$ as a G'-submodule of the H'-module $\mathcal{F}(\mathbb{C}^{n\times N})$ then we have the orthogonal direct sum decomposition

$$\mathcal{I}^{(\lambda)} \cap \mathcal{I}^{(\mu)} = \sum_{\alpha,\beta} \oplus W_{\alpha,\beta}^{(\lambda'_{G'})} , \tag{4.8}$$

where $W_{\alpha,\beta}^{(\lambda'_{G'})}$ is an irreducible G'-module of class (λ), the index α ranges over a set of cardinality equal to $dim[Hom_{G'}(\lambda'_{G'} : \mu'|_{G'})]$ and the index β ranges over a set of cardinality equal to $mult(\mu'_{H'} : R'_{H'})$, the multiplicity of $\mu'_{H'}$ in $R'_{H'}$. By assumption we have

$$\mathcal{I}^{(\lambda)} \approx V^{(\lambda_G)} \otimes W^{(\lambda_{G'})} \quad \text{and} \quad \mathcal{I}^{(\mu)} \approx V^{(\mu_H)} \otimes W^{(\mu'_{H'})} , \tag{4.9}$$

it follows that the dimension of the Hilbert space $W^{(\lambda'_{G'})}$ (see, e.g., [SK], Theorem 8.4) is equal to mult$(\lambda_G : R_G)$ and that the dimension of the Hilbert space $V^{(\mu_H)}$ is equal to mult $(\mu'_{H'} : R'_{H'})$. Now by comparing the dimension of the Hilbert space from Eq. (4.7), Eq. (4.8), and then using Eq. (4.9) we infer that $dim[Hom_H(\mu_H : \lambda|_H)] = dim[Hom_{G'}(\lambda'_{G'} : \mu'|_{G'})]$. □

REMARK 4.2. Theorem 4.1 is very useful when $dim[Hom_H(\mu_H : \lambda|_H)]$ is finite; in particular when one is interested, for example as in Physics, to the explicit decomposition of tensor products of representations. See [Ho4], [KT2], and [KT3] for more discussions of this problem.

The next theorem illustrates the intimate relation between invariant theory and representation theory. This idea of studying representations via invariant theory and vice-versa is ubiquitous in Weyl's book [We] and was extensively investigated by Howe in [Ho1], [Ho2], [Ho3], [Ho5], and [Ho6]; in fact, in [KLT] and in this present article we generalize some examples given in Howe's articles.

Let ρ be an action of a connected Lie group G on $\mathbb{C}^{n\times N}$ then ρ induces a representation R of G on $\mathcal{F}(\mathbb{C}^{n\times N})$ defined by

$$[R(g)f](Z) = f(\rho(g^{-1})Z)$$

Recall that $P(\mathbb{C}^{n\times N})$, the subspace of all polynomial functions on $\mathbb{C}^{n\times N}$, is dense in $\mathcal{F}(\mathbb{C}^{n\times N})$ and a polynomial $p \in P(\mathbb{C}^{n\times N})$ is R (or ρ)-invariant if $R(g)p = p$ for all $g \in G$. Assume that the ring $\mathbb{I}$ of R-invariant polynomials is finitely generated by a set of polynomials $\{p_i, 1 \le i \le r\}$. Let $\mathbb{I}$ act on $\mathcal{F}(\mathbb{C}^{n\times N})$ by multiplication then for $p \in \mathbb{I}$ and $f \in \mathcal{F}(\mathbb{C}^{n\times N})$ we have

$$\begin{aligned}[p(R(g)f)](Z) &= p(Z)f(\rho(g^{-1})Z) \quad \text{and}\\ [R(g)(pf)](Z) &= (pf)(\rho(g^{-1})Z) = p(\rho(g^{-1})Z)f(\rho(g^{-1})Z)\\ &= p(Z)f(\rho(g^{-1})Z)\end{aligned}$$

for all $Z \in \mathbb{C}^{n\times N}$. This means that the action of $\mathbb{I}$ on $\mathcal{F}(\mathbb{C}^{n\times N})$ commutes with R. If $q \in P(\mathbb{C}^{n\times N})$ let $q(D)$ denote the differential operator obtained by replacing $Z_{\alpha j}$ by $\partial/\partial Z_{\alpha j}$ then it can be shown (see, e.g., [TT1], Eq. 2.3) that

$$(4.10)\quad R(g)q(D)R(g^{-1}) = q(D) \text{ for all } g \in G \iff R(g)q = q \text{ for all } g \in G\,,$$

Eq. (4.10) means that an operator $p(D)$ commutes with $R(g)$ if and only if p is an R-invariant polynomial. It follows that together with $dR'(\mathfrak{gl}(n,\mathbb{C}))$, the invariant polynomials and the invariant differential operators $p(D)$, $p \in \mathbb{I}$, generate an associative algebra of operators that commute with R (dR' is the differential of the representation R').

Let $\mathfrak{g}$ denote the Lie algebra of G and dR denote the differential of R which we shall refer to as the infinitesimal action of R. Let $dR_{\mathfrak{g}}$ denote the Lie algebra of operators on $\mathcal{F}(\mathbb{C}^{n\times N})$ generated by the infinitesimal action of R. Together with $dR'(g\ell(n,\mathbb{C}))$ the minimal set of generators $\{p_i, p_i(D), 1 \le i \le r\}$ define a Lie algebra of operators on $\mathcal{F}(\mathbb{C}^{n\times N})$ that commute with R (or equivalently dR). Let $dR'_{\mathfrak{g}'}$ denote this Lie algebra. Our main objective is to find the Lie algebra $\mathfrak{g}'$ and its adjoint group G' and a representation R' of G' which is dual to R. Many examples are given in [KLT] and several more are given in later sections of this present article.

Now let $\mathcal{U}_R \equiv \mathcal{U}(R_G)$ (resp. $\mathcal{U}_{R'} \equiv \mathcal{U}(R'_{G'})$) be the universal enveloping algebra of $dR_{\mathfrak{g}}$ (resp. $dR'_{\mathfrak{g}'}$) then the center $Z(\mathcal{U}_R)$ (resp. $Z(\mathcal{U}_{R'})$ is called the algebra of all *Casimir invariants* of R (resp. R'). Then we have the following important result.

THEOREM 4.3. *Let R and R' be two continuous unitary dual representations of the connected Lie groups G and G', respectively, on the complex Hilbert space $\mathcal{F}(\mathbb{C}^{n\times N})$. Let $\mathcal{W}^{n\times N}$ denote the Weyl algebra defined by the generators $Z_{\alpha i}$, $\frac{\partial}{\partial Z_{\alpha i}}$; $1 \le \alpha \le n$, $1 \le i \le N$, if U_R and $U_{R'}$ are mutual commutants in $\mathcal{W}^{n\times N}$ then the Casimir algebras $\mathcal{Z}(\mathcal{U}_R)$ and $\mathcal{Z}(\mathcal{U}_{R'})$ coincide. Moreover, if $C \in \mathcal{Z}(\mathcal{U}_R)$ then in the isotypic decomposition of $\mathcal{F}(\mathbb{C}^{n\times N}) = \sum_{(\lambda)} \oplus \mathcal{I}^{(\lambda)}$ the restriction of C to each irreducible $R' \otimes R$-submodule $\mathcal{I}^{(\lambda)}$ is a scalar multiple of the identity.*

REMARK 4.4. From the discussion that precedes the theorem it is clear that $\mathcal{U}_{R'}$ is the commutant of $\mathcal{U}_R$ in $\mathcal{W}^{n\times N}$ and in many cases, e.g., Theorems 6.1 and 7.1, a direct verification shows that $\mathcal{U}_R$ is the commutant of $\mathcal{U}_{R'}$ in $\mathcal{W}^{n\times N}$. If G is a reductive group then $\mathcal{Z}(\mathcal{U}_R)$, and hence $\mathcal{Z}(\mathcal{U}_{R'})$, are finitely generated.

PROOF. Since $\mathcal{U}_{R'}$ is the commutant algebra of $\mathcal{U}_R$ in the Weyl algebra $\mathcal{W}^{n\times N}$ it is clear that $\mathcal{Z}(\mathcal{U}_R) = \mathcal{U}_R \cap \mathcal{U}_{R'}$. By symmetry $\mathcal{Z}(U_{R'}) = \mathcal{U}_{R'} \cap \mathcal{U}_R$. So $\mathcal{Z}(\mathcal{U}_{R'}) = \mathcal{Z}(\mathcal{U}_{R'})$. Now let $C \in \mathcal{Z}(\mathcal{U}_{R'}) = \mathcal{Z}(\mathcal{U}_R)$ and consider $\mathcal{F}(\mathbb{C}^{n\times N})$ as a $G' \times G$-module. Then by hypothesis the continuous unitary representation $R \otimes R'$ of $G' \times G$ on $\mathcal{F}(\mathbb{C}^{n\times N})$ decomposes into irreducible isotypic components of multiplicity one of the form $\mathcal{F}(\mathbb{C}^{n\times N}) = \sum_{(\lambda)} \oplus \mathcal{I}^{(\lambda)}$. Since C is a continuous endomorphism of $\mathcal{I}^{(\lambda)}$ that commutes with the restriction of $R' \otimes R$ to $\mathcal{I}^{(\lambda)}$ it follows from a generalized version of Schur's lemma (cf., e.g. [Br] Theorem 4.6 or [Na]) that the restriction of C to each irreducible $R' \otimes R$-submodule $\mathcal{I}^{(\lambda)}$ is a scalar multiple of the identity. □

In the next two sections we shall apply the procedure and theorem stated above to specific examples.

5. The Harmonic Representations of $U(N)$

In this section we describe the harmonic representations of $U(N)$ or equivalently of $GL(N,\mathbb{C})$. This result which we will use in Section 6 is just a slight generalization of several theorems in [TT2], so we will not repeat the proof (with its obvious modifications) and refer the interested reader to [TT2]. Let p, q, and n be positive integers such that $p+q = n$. In this section we make the additional assumption that $N \geq \max(p, q)$.

Let E (resp. F) denote the complex vector space of all $N \times q$ (resp. $p \times N$) matrices. Then $GL(N,\mathbb{C})$ acts on $F \times E$ via the mapping

$$((\xi, x), g) \to (\xi, x) \cdot g \equiv (\xi g, g^{-1}x), \quad (\forall (\xi, x) \in F \times E) .$$

The bilinear form $(\cdot|\cdot)$ defined by the equation

$$(5.1) \quad ((\xi_1, x_1)|(\xi_2, x_2)) = Tr(\xi_1 x_2) + Tr(\xi_2 x_1), \ (\forall (\xi_1, x_1), (\xi_2, x_2) \in F \times E) ,$$

is clearly symmetric, nondegenerate and invariant under the action of $GL(N,\mathbb{C})$ defined above. To every element $(\xi, x) \in F \times E$ corresponds a linear form $(\xi, x)^*$ defined by

$$(\xi, x)^*((\xi', x')) = ((\xi, x)|(\xi', x')), \ (\forall (\xi', x') \in F \times E) .$$

The correspondence $(\xi, x) \to (\xi, x)^*$ is clearly an isomorphism of the vector space $F \times E$ onto its dual. Let $P \equiv P(F \times E)$ denote the algebra of all polynomial functions on $F \times E$. Then the algebras $P(F)$ and $P(E)$ of polynomial functions on F and E, respectively, can be considered as subalgebras of P. Each linear form $(\xi, x)^*$ can be considered as a homogeneous polynomial of degree 1 in P.

Define an action of $GL(N,\mathbb{C})$ on P by

$$(g\cdot f)(\xi,x)=f(\xi g,g^{-1}x)\ \ (\forall(\xi,x)\in F\times E)\ .$$

If $p\in P$ we let $p(D)$ denote the differential operator obtained by substituting the partial derivatives $\partial/\partial\xi_{ij}$ and $\partial/\partial x_{ij}$ for the variables ξ_{ij},x_{ij} in p. It is well-known that the map $p\to p(D)$ is an isomorphism of P into the symmetric algebra of differential operators on P with constant coefficients. The group $GL(N,\mathbb{C})$ acts on this algebra via $(g,p(D))\to g\cdot p(D)$ defined by

$$(g\cdot p(D))f=g\cdot[p(D)(g^{-1}\cdot f)],\ \ (\forall f\in P)\ .$$

A polynomial $p\in P$ (resp. a differential operator $p(D)$) is *invariant* with respect to the action of $GL(N,\mathbb{C})$ if $g\cdot p=p$ (resp. $g\cdot p(D)=p(D)$). One can easily verify that p is invariant if and only if $p(D)$ is invariant. Let $\mathbb{I}$ denote the subalgebra of P consisting of all invariant polynomials p. Let $\mathbb{I}^+$ denote the subset of $\mathbb{I}$ which consists of all $j\in\mathbb{I}$ such that $j(0,0)=0$.

A polynomial function $f\in P$ is said to be *harmonic* if $j(D)f=0$ for all $j\in\mathbb{I}^+$. It is clear that the harmonic polynomial functions form a subspace of P. From the invariant theory (cf. [We]) it follows that $\mathbb{I}$ is generated by the constants and by the algebraically independent polynomial functions defined by

$$p_{ij}(\xi,x)=\sum_{\ell=1}^{N}\xi_{i\ell}x_{\ell j}\ ,\quad 1\le i\le p\ ,\quad 1\le j\le q\ . \tag{5.2}$$

Let H denote the subspace (of P) of all harmonic polynomial functions, then obviously

$$H=\{f\in P: p_{ij}(D)f=0\quad \forall i,j;1\le i\le p,\ 1\le j\le q\}\ .$$

Let $\mathbb{I}^+P$ denote the ideal in P generated by $\mathbb{I}^+$ and V denote the null cone of all (ξ,x) such that $f(\xi,x)=0$ $(\forall f\in\mathbb{I}^+P)$ then obviously

$$V=\{(\xi,x)\in F\times E:\xi x=0\}\ . \tag{5.3}$$

Let $H(V)$ denote the subspace of H spanned by all polynomial functions of the form $[(\xi,x)^*]^m$ for all (ξ,x) in V and all integers $m\ge 0$. Let $P|_V$ denote the algebra of all restrictions $f|_V$, $f\in P$ then

THEOREM 5.1. *Under the assumptions and notations above the following statements hold:*

(i) The ideal $\mathbb{I}^+P$ is prime.

(ii) $P=\mathbb{I}^+P\oplus H$ and $H=H(V)$.

(iii) The mapping $j\otimes f\to jf$ of $\mathbb{I}\otimes P$ onto P is an isomorphism.

(iv) The restriction map $f\to f|_V$ of H onto $P|_V$ is an isomorphism of $GL(N,\mathbb{C})$-modules.

Let B_q (resp. B_p) denote the group of all lower triangular matrices of order q (resp. order p). If $(r_1,\dots,r_p)$ and $(s_1,\dots,s_q)$ are two tuples of integers such

that $0 \leq s_1 \leq s_2 \leq \ldots \leq s_q$ and $r_1 \geq \ldots \geq r_p \geq 0$ we define the holomorphic characters $\lambda_r : B_p \to \mathbb{C}^*$ and $\lambda_s : B_q \to \mathbb{C}^*$ by $\lambda_r(b_p) = b_{11}^{r_1} \ldots b_{pp}^{r_p}$, $(\forall b_p \in B_p)$ and $\lambda_s(b_q) = b_{11}^{s_1} \ldots b_{qq}^{s_q}$, $(\forall b_q \in B_q)$. Set

$$P(E, \lambda_q) = \{f \in P(E) : f(xb_q) = \lambda_q(b_q) f(x);\ \forall (x, b_q) \in E \times B_q\}$$
$$P(F, \lambda_p) = \{f \in P(F) : f(b_p \xi) = \lambda_p(b_p) f(\xi);\ \forall (\xi, b_p) \in F \times B_p\}$$

Let $H^{(\lambda)}$ denote the subspace of H generated by the elements $f \in P(E, \lambda_p) \otimes P(F, \lambda_q)$ which also satisfy the condition $p_{ij}(D)f = 0$, $1 \leq i \leq p$, $1 \leq j \leq q$. Let $R^{(\lambda)}$ denote the representation of $GL(N, \mathbb{C})$ on $H^{(\lambda)}$ defined by $(R^{(\lambda)}(g)f)(\xi, x) = f(\xi g, g^{-1}x) \equiv (g \cdot f)(\xi, x)$. Then we have

THEOREM 5.2. *The representation $R^{(\lambda)}$ of $GL(N, \mathbb{C})$ on $H^{(\lambda)}$ is an irreducible holomorphic representation of class (λ) which has signature*

$$\underbrace{(r_1, \ldots, r_p, 0, \ldots, 0, -s_1, -s_2, \ldots, -s_q)}_{N} .$$

Moreover, its restriction to $U(N)$ is also irreducible with the same signature.

6. The Oscillator Dual Representations of the Pair $(U(p, q), U(N))$

Let p, q and n be integers such that $p + q = n$. Let G denote the group $U(N)$ and let $\mathfrak{g}$ denote its Lie algebra. Set $G_{\mathbb{C}} = GL(N, \mathbb{C})$ and partition every matrix $Z \in \mathbb{C}^{n \times N}$ into

$$Z = \begin{matrix} p \\ q \end{matrix} \overbrace{\begin{bmatrix} Z_1 \\ \cdots \\ Z_2 \end{bmatrix}}^{N} .$$

Consider the Fock space $\mathcal{F}(\mathbb{C}^{n \times N})$ and identify the vector space $F \times E$ in Section 5 with $\mathbb{C}^{n \times N}$ via the mapping $(\xi, x) \to Z$, with $Z_1 = \xi$ and $Z_2 = x^T$, for all $(\xi, x) \in F \times E$. Then clearly the action of $GL(N, \mathbb{C})$ on $P(F \times E)$ defined in Section 5 becomes

$$[R^G(g)f](Z) \equiv (g \cdot f)(Z) = f\left(\begin{bmatrix} Z_1 g \\ Z_2 g^{\vee} \end{bmatrix}\right), \quad \text{where } g^{\vee} = (g^{-1})^T$$

for all $g \in G_{\mathbb{C}}$ and $f \in \mathcal{F}(\mathbb{C}^{n \times N})$. It follows that the algebra of all $G_{\mathbb{C}}$ (or G)-invariant polynomials are generated by the constants and the pq algebraically independent polynomials

$$(6.1) \qquad f_{i,p+j}(Z) = (Z_1 Z_2^T)_{ij} = \sum_{s=1,\ldots,N} Z_{is} Z_{p+j,s} , \quad 1 \leq i \leq p , \quad 1 \leq j \leq q ,$$

The infinitesimal action of $GL(n, \mathbb{C})$ on $\mathcal{F}(\mathbb{C}^{n \times N})$ is given by

$$(6.2) \qquad L_{ij} = \sum_{s=1,\ldots,N} Z_{is} \frac{\partial}{\partial Z_{js}} , \quad 1 \leq i, j \leq n ,$$

Set

$$D_{i,p+j} = f_{i,p+j}(D) = \sum_{s=1,\dots,N} \frac{\partial^2}{\partial Z_{is}\partial Z_{p+j,s}} \tag{6.3}$$

and denote by $P_{i,p+j}$ the operator defined on $\mathcal{F}(\mathbb{C}^{n\times N})$ by multiplication by the polynomial $f_{i,p+j}$. Then an easy computation shows that

$$-[P_{i,p+j}, D_{i',p+j'}] = \delta_{ii'}L_{p+j,p+j'} + \delta_{jj'}L_{i,i'} + N\delta_{ii'}\delta_{jj'}$$

Let $(\mathfrak{g}'_{\mathbb{C}})^0$ denote the Lie algebra $s\ell(n,\mathbb{C})$ with the standard basis $\{H_i, E_{ij,i\neq j}\}$ where the H_i's form a basis for a Cartan subalgebra of $(\mathfrak{g}'_{\mathbb{C}})^0 (1 \leq i \leq n-1)$ and if we set $X_i = E_{i,i+1}$ and $Y_i = E_{i+1,i}$ then $(\mathfrak{g}'_{\mathbb{C}})^0$ can be defined by the generators $X_i, Y_i, H_i; 1 \leq i \leq n-1$, which satisfy the Cartan-Weyl relations. Define a representation of $(\mathfrak{g}'_{\mathbb{C}})^0$ or $\mathfrak{su}(p,q)$ in $\mathcal{F}(\mathbb{C}^{n\times N})$ sending the basis elements of $(\mathfrak{g}'_{\mathbb{C}})^0$ to the operators defined by Eq. (6.1), (6.2) and (6.3) according to the following scheme

(6.4)

$$\begin{cases} H_i & \to L_{ii} - L_{i+1,i+1}, 1 \leq i \leq p-1 \\ H_p & \to L_{p,p} + L_{p+1,p+1} + N \\ H_{p+j} & \to -L_{p+j,p+j} + L_{p+j+1,p+j+1} = -(L_{p+j,p+j} - L_{p+j+1,p+j+1}), \\ & \quad 1 \leq j \leq q-1 \\ E_{i,j} & \to L_{i,j}, 1 \leq i,j \leq p, i \neq j \\ E_{i,p+j} & \to P_{i,p+j}, 1 \leq i \leq p, 1 \leq j \leq q \\ E_{p+j,i} & \to -D_{i,p+j}, 1 \leq i \leq p, 1 \leq j \leq q \\ E_{p+i,p+j} & \to -L_{p+j,p+i}, 1 \leq i,j \leq q, i \neq j \end{cases}$$

Equation (6.4) extends to a faithful representation of $(\mathfrak{g}'_{\mathbb{C}})^0$ or $\mathfrak{su}(p,q)$ into $\mathcal{F}(\mathbb{C}^{n\times N})$. Actually we can view this representation as dual to a tensor product representation as follows:

Consider the tensor product representation of $G_{\mathbb{C}}\times G_{\mathbb{C}}$ (or $G\times G$) on $\mathcal{F}(\mathbb{C}^{n\times N})$ defined by

$$[(R_1 \otimes R_2^{\checkmark})(g_1,g_2)f]\left(\begin{bmatrix} Z_1 \\ \cdots \\ Z_2 \end{bmatrix}\right) = f\left(\begin{bmatrix} Z_1 g_1 \\ \cdots \\ Z_2 g_2^{\checkmark} \end{bmatrix}\right)$$

then the representation R^G is just the restriction of $R_1 \otimes R_2^{\checkmark}$ to the diagonal subgroup $G_{\mathbb{C}}$ of $G_{\mathbb{C}}\times G_{\mathbb{C}}$. This problem was studied in [KT3] for tensor products of the form $R_1 \otimes R_2$ but the case $R_1 \otimes R_2^{\checkmark}$, where $R_2^{\checkmark}$ is the contragredient representation to R_2, can be handled in exactly the same way. Thus if we let

$$R_{\alpha\beta} = \sum_{k=1}^{p} (Z_1)_{k\alpha}\frac{\partial}{\partial (Z_1)_{k\beta}}, R'_{\alpha\beta} = -\sum_{k=1}^{q} (Z_2)_{k\beta}\frac{\partial}{\partial (Z_2)_{k\alpha}}, 1 \leq \alpha,\beta \leq N \tag{6.5}$$

and let [R], [R'] denotes the matrices with (α,β)-entry equal to $R_{\alpha\beta}$ and $R'_{\alpha\beta}$, respectively, then we know (see [KT3] Prop. 3.3) that the R^G-Casimir invariants

are generated by the differential operators of the form

(6.6) $\qquad Tr([R]^{d_1}[R']^{d_2})$ where d_1 and d_2 are integers ≥ 0 .

The infinitesimal of the dual action to R^G, which we shall denote by $L^{G'}$, where G' denote the group

$$U(p,q) = \left\{ g \in GL(n,\mathbb{C}) : g \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} g^* = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \right\} ,$$

is given by the basis $\{L_{i,j}, -L_{p+i',p+j'}, P_{i,p+k}, -D_{i,p+k}\}$, $1 \leq i,j \leq p$, $1 \leq i',j' \leq q$, $1 \leq k \leq q$. The global representation $L^{G'}$ is explicitely in [KV] on the Schrödinger space $L^2(\mathbb{R}^{n\times N})$ and to obtain the representation $L^{G'}$ on the Fock space $\mathcal{F}(\mathbb{C}^{n\times N})$ one must use the unitary Bargmann-Segal transform from the Schrödinger space onto the Fock space; the final form of this representation is quite complicated and we refer the interested reader to [DES] and [HD]. Let X, Y, P, D denote the matrices whose entries are L_{ij}, $L_{p+i',p+j'}$, $P_{i,p+k}$, and $D_{i,p+k}$, respectively and set

(6.7) $$[L^{G'}] = \begin{bmatrix} X & \vdots & P \\ \cdots & \cdots & \cdots \\ -D^T & \vdots & -Y^T \end{bmatrix}$$

Note that the L_{ij} and $-L_{p+i',p+j'}$ generate over $\mathbb{C}$ a subalgebra $\mathcal{T}_{\mathbb{C}} \approx \mathfrak{gl}(p,\mathbb{C}) \oplus \mathfrak{gl}(q,\mathbb{C})$, the $P_{i,p+k}$ generate a subalgebra $\mathfrak{p}_+$ and the $D_{i,p+k}$ generate a subalgebra $\mathfrak{p}_-$ in such a way that $\mathcal{T}_{\mathbb{C}} + \mathfrak{p}_+ + \mathfrak{p}_-$ gives a realization of the Lie algebra $\mathfrak{gl}(n,\mathbb{C})$.

THEOREM 6.1. *The algebra of Casimir invariants of the representation $L^{G'}$ is generated by the constants and n algebraically independent non-commutative differential operators $Tr([L^{G'}]^i)$, $1 \leq i \leq n$ if $n \leq N$, and $1 \leq i \leq N$ if $n > N$, where $[L^{G'}]$ is given by Eq. (6.7).*

PROOF. The L_{ij}, $-L_{p+i',p+j'}$, $P_{i,p+k}$, $-D_{i,p+k}$ generate over $\mathbb{R}$ the Lie algebra $\mathfrak{u}(p,q)$ whose complexification is $\mathfrak{gl}(n,\mathbb{C})$. It follows from our general remarks in Section 3 that a set of generators of the $L^{G'}$ Casimir invariants can be obtained from a set of generators of the Casimir invariants of the action of $\mathfrak{gl}(n,\mathbb{C})$ on $\mathcal{F}(\mathbb{C}^{n\times N})$. And it is well-known that the constants and the differential operators $Tr([L^G]^i)$, $1 \leq i \leq n$, generate the Casimir invariants for $\mathfrak{gl}(n,\mathbb{C})$. □

7. The Oscillator Dual Representations of the Pair $(U(p,q) \ltimes_\tau H_n, U(N-1))$

Set $G = U(N)$ and let K denote the subgroup of G consisting of all matrices

of the form

$$\begin{pmatrix} \gamma & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & 1 \end{pmatrix}, \ \gamma \in U(N-1) \ .$$

If R^K denotes the restriction of R^G to K, then a system of generators of the infinitesimal action of $\tilde{R} \equiv R^{K_{\mathbb{C}}}$ is given by

$$(7.1)\quad \tilde{R}_{ab} = \sum_{k=1}^{p} (Z_1)_{ka} \frac{\partial}{\partial (Z_1)_{kb}} \ , \ \tilde{R}'_{ab} = -\sum_{k=1}^{q} (Z_2)_{kb} \frac{\partial}{\partial (Z_2)_{ka}} \ , \ 1 \le a, b \le N-1$$

The dual representation of $\tilde{R}$ is defined as follows. In addition to the polynomial functions $f_{i,p+j}$ defined by Eq. (6.1), which are obviously $\tilde{R}$-invariant, the column vector $\begin{bmatrix} Z_{1N} \\ \vdots \\ Z_{nN} \end{bmatrix}$ is unaffected by $\tilde{R}$ so that the linear polynomial function Z_{iN}, $1 \le i \le n$, also generate $\tilde{R}$-invariant polynomials. The system $\{Z_{iN}, \frac{\partial}{\partial Z_{iN}}, 1 \le i \le n, \text{ and } I\}$, where I denotes the identity operator on $\mathcal{F}(\mathbb{C}^{n \times N})$, generates a Heisenberg algebra of rank n. The infinitesimal dual action of R^K is a representation of the semidirect product (sum) of $\mathfrak{u}(p,q)$ and $\mathcal{H}_n$ on $\mathcal{F}(\mathbb{C}^{n \times N})$ which is generated by the following operators

$$(7.2)\quad \begin{cases} L_{i,j}, -L_{p+i',p+j'}, P_{i,p+k}, -D_{i,p+k} & \text{as given by Eq. (6.7)} \\ Z_{iN}, \frac{\partial}{\partial Z_{iN}}, 1 \le i \le n & \text{and } I \end{cases}$$

And we have the following commutation relations

$$[L_{i,j}, Z_{kN}] = \delta_{jk} Z_{iN}, [L_{ij}, \frac{\partial}{\partial Z_{kN}}] = -\delta_{ik} \frac{\partial}{\partial Z_{jN}} \ ,$$

$$[-L_{p+i',p+j'}, Z_{kN}] = -\delta_{p+j',k} Z_{p+i',N}, [-L_{p+i',p+j'}, \frac{\partial}{\partial Z_{kN}}] = \delta_{p+i',k} \frac{\partial}{\partial Z_{p+j',N}} \ ,$$

$$[P_{i,p+k}, Z_{\ell N}] = 0, [-D_{i,p+k}, Z_{\ell_N}] = \delta_{p+k,\ell} \frac{\partial}{\partial Z_{iN}} - \delta_{i\ell} \frac{\partial}{\partial Z_{p+\ell,N}} \ ,$$

$$[P_{i,p+k}, \frac{\partial}{\partial Z_{\ell N}}] = -\delta_{i\ell} Z_{p+k,N} - \delta_{p+k,\ell} Z_{iN}, [-D_{i,p+k}, \frac{\partial}{\partial Z_{\ell N}}] = 0 \ .$$

The global action which is dual to R^K is a representation of the semidirect product $U(p,q) \ltimes H_n$ on $\mathcal{F}(\mathbb{C}^{n \times N})$, where H_n is a Heisenberg group of rank n. Set $K' = U(p,q) \ltimes H_n$ and let $L^{K'}$ denote this dual representation, then $L^{K'}$ can be explicitly defined as in [KLT].

Let

$$(7.3)\quad \begin{cases} \tilde{L}_{ij} & = \sum_{s=1,\dots,N-1} Z_{is} \frac{\partial}{\partial Z_{js}}, 1 \le i, j \le p \ , \\ \tilde{L}_{p+i',p+j'} & = \sum_{s=1,\dots,N-1} Z_{p+i',p+j'}, 1 \le i', j' \le q \ , \\ \tilde{P}_{i,p+k} & = \sum_{s=1,\dots,N-1} Z_{is} Z_{p+j,s}, 1 \le i \le p, 1 \le k \le q \ , \\ \tilde{D}_{i,p+k} & = \sum_{s=1,\dots,N-1} \frac{\partial^2}{\partial Z_{is} \partial Z_{p+j,s}}, 1 \le i \le p, 1 \le k \le q \ . \end{cases}$$

Let $\tilde{X}, \tilde{Y}, \tilde{P}, \tilde{D}$ denote the matrices whose entries a $\tilde{L}_{ij}$, $\tilde{L}_{p+i',p+j'}$, $\tilde{P}_{i,p+k}$, and $\tilde{D}_{i,p+k}$, respectively, as given by Eq. (7.3) and set

$$[\tilde{L}^{K'}] = \begin{bmatrix} \tilde{X} & \vdots & \tilde{P} \\ \cdots & \cdots & \cdots \\ -\tilde{D}^T & \vdots & -\tilde{Y}^T \end{bmatrix},$$

Then we have the following.

THEOREM 7.1. *The algebra of all Casimir invariant differential operators of the representation $L^{K'}$ of the semidirect product $U(p,q) \ltimes H_n$, which is dual to the representation R^K of $K \approx U(N-1)$, on $\mathcal{F}(\mathbb{C}^{n\times N})$ is generated by*

(i) the constants and the n algebraically independent Casimir operators $Tr([\tilde{L}^{K'}]^i)$, $1 \leq i \leq n$, if $n \leq N$.

(ii) the constants and the N algebraically independent Casimir operators $Tr([\tilde{L}^{K'}]^i)$, $1 \leq i \leq N$ if $N < n$.

PROOF. Clearly we have the isomorphism $\mathcal{F}(\mathbb{C}^{n\times N}) \approx \mathcal{F}(\mathbb{C}^{n\times(N-1)}) \otimes \mathcal{F}(\mathbb{C}^{n\times 1})$ so that under the restriction of R^G to K the representation R^K can be considered as the representation $\tilde{R}\otimes I$, where $\tilde{R}$ is the representation of $U(N-1)$ on $\mathcal{F}(\mathbb{C}^{n\times(N-1)})$ and I is the identity representation of $U(1)$ on $\mathcal{F}(\mathbb{C}^{n\times 1})$. We have the orthogonal direct sum decomposition into isotypic components (cf. [KV] or [Ho2])

$$\mathcal{F}(\mathbb{C}^{n\times(N-1)}) = \sum_{(\mu)} \oplus \mathcal{I}^{(\mu)}(\mathbb{C}^{n\times(N-1)}) \tag{7.4}$$

where (μ) denote both a signature of an irreducible $U(N-1)$ submodule and an irreducible $U(p,q)$-submodule ((μ) is of the form $(r_1,\ldots,r_p,-s_1,\ldots,-s_q)$ where $r_1,\ldots,r_p$ and $s_1,\ldots,s_q$ are integers such that $r_1 \geq \ldots \geq r_p \geq 0$ and $0 \leq s_1 \leq \ldots \leq s_q$ if $n = N-1$ for example). It can be shown (cf. [KLT]) that the representation $L^{K'}$ of the semidirect product $U(p,q) \ltimes H_n$ on $\mathcal{F}(\mathbb{C}^{n\times N}) \approx \mathcal{F}(\mathbb{C}^{n\times(N-1)}) \otimes \mathcal{F}(\mathbb{C}^{n\times 1})$ is unitary so that we also have the orthogonal direct sum decomposition into isotypic components

$$\mathcal{F}(\mathbb{C}^{n\times N}) = \sum_{(\mu)} \oplus (\mathcal{I}^{(\mu)}(\mathbb{C}^{n\times N-1}) \otimes \mathcal{F}(\mathbb{C}^{n\times 1})),$$

since H_n acts irreducibly on $\mathcal{F}(\mathbb{C}^{n\times 1})$. From [KT3] Prop. 3.3 we know that the Casimir invariants of $\tilde{R}$ and of its dual representation coincide. But since $L^{K'}$ and R^K are dual representations and obviously the R^K-Casimir invariants are the same as the $\tilde{R}$-Casimir invariants, Theorem 4.3 implies that the $L^{K'}$-Casimir invariants coincide with the $\tilde{L}$-Casimir invariants. The statements i) and ii) of the theorem follow from Theorem 6.1. □

8. Conclusion

In this article we have proven several general theorems and given several examples of dual representations together with their Casimir invariant differential operators. Let us conclude by describing a general procedure for finding the dual representation of a representation R^G of a connected Lie group G on $\mathcal{F}(\mathbb{C}^{n\times N})$. Suppose that G is a connected Lie group and R^G is a completely reducible unitary representation of G in $\mathcal{F}(\mathbb{C}^{n\times N})$ (this situation includes the case R^G is the restriction to the diagonal subgroup of an m-fold tensor product of representations of G). If the identity representation of G occurs as a R^G-submodule and if this submodule is a space of polynomial functions then by definition these polynomial functions are the R^G-invariants. If the algebra generated by these R^G-invariant polynomials is finitely generated (this is the case, for example, when G is reductive) we may find a set of generators; in concrete cases, these may be obtained by computing the Clebsch-Gordan coefficients of the identity representations. Let $\{p_i\}$ denote these generators and use the set $\{p_i, p_i(D)\}$ to find the dual of dR^G, and then, if possible, obtain the group dual action. To illustrate this procedure let us consider a simple example, disregarding its well-known results which will be derived subsequently.

Let R denote the representation of $GL(N,\mathbb{C})$ (or $U(N)$), $N > 2$, on the Fock space $\mathcal{F}(\mathbb{C}^{1\times N})$ defined by

$$R(y)f(Z) = f(Zy), (\forall f \in \mathcal{F}(\mathbb{C}^{1\times N}), y \in GL(N,\mathbb{C})) \ .$$

Then $\mathcal{F}(\mathbb{C}^{1\times N})$ is decomposed into an orthogonal direct sum decomposition

$$\mathcal{F}(\mathbb{C}^{1\times N}) = \sum_{m=0,1,2,\ldots} \oplus P^m(\mathbb{C}^{1\times N}) \ ,$$

where $P^m(\mathbb{C}^{1\times N})$ denote the vector space of all homogeneous polynomials of degree m in N variables $Z_1, \ldots, Z_N$. The dual representation to R is the representation L of $GL(1,\mathbb{C})$ which is obtained by left translation on $\mathcal{F}(\mathbb{C}^{1\times N})$. Each $P^m(\mathbb{C}^{1\times N})$ is therefore an isotypic component. The submodule $P^0(\mathbb{C}^{1\times N}) \approx \mathbb{C}$ is the identity component of R. Thus the only R-invariants are the scalars which is the uninteresting case. Now let $G = SO(N)$ and denote the restriction of R to G by R^G. Each R^G-submodule $P^m(\mathbb{C}^{1\times N})$ can be further decomposed into irreducible R^G-submodules as

$$P^m(\mathbb{C}^{1\times N}) = \sum_{i=0,\ldots,[m/2]} \oplus p^i \mathcal{H}^{m-2i} \ , \tag{8.1}$$

where in Eq. (8.1) $p(Z) = \sum_{j=1}^N Z_j^2$ and $\mathcal{H}^{m-2i}$ is the space of all harmonic polynomials of degree $m-2i$, $1 \le i \le [m/2]$. Thus we have the spectral decomposition of the R^G-submodule $P^m(\mathbb{C}^{1\times N})$ given in terms of signatures of

irreducible representations of G as

$$(8.2) \qquad \underbrace{(m,0,\dots,0)}_{[N/2]} \oplus \underbrace{(m-2,0,\dots,0)}_{[N/2]} \oplus \cdots \oplus \underbrace{(m-2[m/2],0,\dots,0)}_{[N/2]} .$$

In Eq. (8.2) we see that the identity representation (signature $(0,0,\dots,0)$) occurs when m is even. Therefore the R^G-submodule which corresponds to the identity subrepresentation is the algebra of all polynomial functions generated by $\{p^{\frac{m}{2}},\ m=0,2,4\cdots\}$. This is precisely the algebra of R^G-invariants which is generated by p. Thus if we set $P=-p$, $D=p(D)$ and $H=4\sum_{j=1}^{n} Z_j\frac{\partial}{\partial Z_j}+2N$ then $\{P,D,H\}$ generates the infinitesimal dual action to R^G. Clearly $\{P,D,H\}$ forms a basis of $\mathfrak{sl}(2,\mathbb{R})$ (or $\mathfrak{sp}(2,\mathbb{R})$), and this infinitesimal action eventually leads to the group dual action (see, e.g., [KLT] for details).

Finally let us remark that in this article we concentrated on the pair $(U(p,q), U(N))$ since this dual action arises naturally from the harmonic representations of $U(N)$ studied in Section 5, but if we restrict those representations of $U(N)$ to a subgroup such as $O(N)$ or $Sp(N)$ (N even) then we will obtain many equally interesting dual representations.

REFERENCES

[Br] F. Bruhat, *Représentations des Groupes Localement Compacts*, Lecture Notes, Ecole Normale Supérieure, Paris, France, 1971.

[Ch] C. Chevalley, *Théorie des Groupes de Lie*, Hermann, Paris, France, 1955.

[DES] M. Davidson, T. Enright and R. Stanke, *Differential operators and highest weight representations*, Memoirs of the AMS **94** (1991).

[Di] J. Dixmier, *Algèbres enveloppantes*, Gauthier-Villars, Paris, France, 1974.

[EHW] T. Enright, R. Howe and N. Wallach, *A classification of unitary highest weight modules*, Prog. Math. 40 (1983), 97–143, Proceedings, Representation Theory of Reductive Groups, (Park City, UT, 1982).

[Ge] S. Gelbart, *Examples on dual reductive pairs, automorphic forms, representations and L- functions, Part I* (A. Borel and W. Casselman, eds.), Proc. Symp. Pure Math, vol. 33, Amer. Math. Soc., Providence, RI, 1979, pp. 287–96.

[Go] R. Godement, *Introduction ā la théorie des groupes de Lie*, Publication Math. de l'Université, Paris VII, France, 1982.

[GK1] K. Gross and R. Kunze, *Fourier Bessel transforms and holomorphic discrete series*, Proc. of the Maryland Conf. on Harmonic Analysis 1971, Lec. Notes in Math 266, Springer, Berlin, Heidelberg, New York, 1972, pp. 79–122.

[GK2] K. Gross and R. Kunze, *Bessel Functions and representation* I, J. Funct. Anal. **22** (1976), 73–105; II, J. Funct. Anal. **25** (1977), 1–49.

[HD] H. Ding, *Operator-valued Bessel functions, holomorphic discrete series and harmonic representations of $U(p,q)$*, J. Funct. Anal. **108** (1992), 187–230.

[Ho1] R. Howe, *Remarks on classical invariant theory*, Trans. A.M.S. **313** no. No. 2 (1989), 539–570.

[Ho2] R. Howe, *Dual pairs in physics: Harmonic oscillators, photons, electrons, and singletons*, Lectures in Appl. Math., vol. 21, Amer. Math. Soc., Providence, RI, 1985, pp. 179–206.

[Ho3] R. Howe, *Reciprocity laws in the theory of dual pairs*, Progr. Math. 40 (1983), 159–175, Proceedings, Representation Theory of Reductive Groups (Park City, UT, 1982).

[Ho4] R. Howe, *$(GLn; GLm)$-duality and symmetric plethysm*, Proc. Ind. Acad. Sci., Math. Sci. **97** no. No. 1–3 (1987), 85–109.

[Ho5] R. Howe, *The classical groups and invariants of binary forms*, Proc. Symp. in Pure Math. **48** (1988), 133–166.

[Ho6] R. Howe, *Transcending the classical invariant theory*, J. A.M.S. **2** no. No. 3 (1989), 535–552.

[KLT] W. H. Klink, E. Y. Leung and T. Ton-That, *Casimir operators of semidirect products of semisimple with Heisenberg groups*, (submitted for publication).

[KT1] W. H. Klink and T. Ton-That, *Duality in representation theory*, Ulam Quarterly **1** (1992), 41–49.

[KT2] W. H. Klink and T. Ton-That, *n-fold tensor products of $GL(N, \mathbb{C})$ and decomposition of Fock spaces*, J. Funct. Anal. **84** (1989), 1–18.

[KT3] W. H. Klink and T. Ton-That, *On resolving the multiplicity of arbitrary tensor products of $U(N)$ groups*, J. Phys. A: Math. Gen. **21** (1988), 3877–3892.

[KV] M. Kashiwara and M. Vergne, *On the Segal-Shale-Weil representation and harmonic polynomial*, Invent. Math. **44** (1978), 1–47.

[MQ] M. Moshinsky and C. Quesne, *Noninvariance groups in the second quantization picture and their applications*, J. Math. Phys. **11** (1970), 1631–1639.

[Na] M. A. Naimark, *Normed Rings*, Wolters-Noordhoff, Groningen, The Netherlands, 1970.

[Ni] K. Nishiyama, *Decomposing oscillator representations of $osp(2n/n; \mathbb{R})$ by a super dual pair $osp(2/1; \mathbb{R}) \times so(n)$*, Composito Math. **80** (1991), 137–149.

[Ra] G. Racah, *Group theory and spectroscopy*, Ergebn. der Exakten Nuturwiss. **37** (1965), 28–84.

[Sa] M. Saito, *Représentations unitaires des groupes symplectiques*, J. Math. Soc. Japan **24** (1972), 232–251.

[Sc] I. Schur, *dissertation*, Berlin, 1901.

[SK] I. Segal and R. Kunze, *Integrals and operators*, 2nd edition, Springer-Verlag, Berlin, Heidelberg, New York, 1978.

[TT1] T. Ton-That, *Lie group representations and harmonic polynomials of a matrix variable*, Trans. Amer. Math. Soc. **216** (1976), 1–46.

[TT2] T. Ton-That, *Sur la décomposition des produits tensoriels des représentations irréductibles de $GL(k, \mathbb{C})$*, J. Math. Pures et Appl. **56** (1977), 251–262.

[VK] N. Ja. Vilenkin and **A.** V. Klymik, *Representation of Lie groups and special functions* Vol. 2, Kluwer Acad. Publ., Dordrecht, Boston, London, 1993, pp. 61.

[We] H. Weyl, *The classical groups, their invariants and representations*, 2nd ed., Princeton Univ. Press, Princeton, NJ, 1946.

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On the Harish-Chandra Homomorphism of Invariant Differential Operators on a Reductive Lie Algebra

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Dedicated to Ray A. Kunze

ABSTRACT. Harish-Chandra defined an algebra homomorphism from the polynomial differential operators on a complex reductive Lie algebra that are invariant under the adjoint group into the polynomial differential operators on a Cartan subalgebra that are invariant under the Weyl group. We give a formula for this homomorphism analogous to the Harish-Chandra homomorphism from the center of the universal enveloping algebra onto the Weyl group invariants in the symmetric algebra of a Cartan subalgebra. Our construction involves the analysis of a certain $\mathcal{D}$-module on the Lie algebra which is of independent interest.

Introduction

If V is a finite-dimensional vector space over $\mathbb{C}$ then we denote by $\mathcal{D}(V)$ the algebra of all differential operators on V with polynomial coefficients. If G is a subgroup of $\mathrm{GL}(V)$ then we denote by $\mathcal{D}(V)^G$ the subalgebra of all elements of $\mathcal{D}(V)$ that commute with the action of G on functions. In this article we consider the case when $V = \mathfrak{g}$ is a reductive Lie algebra over $\mathbb{C}$. Let G denote the identity component of the group of automorphisms of $\mathfrak{g}$ that act by the identity on the center of $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let W denote the Weyl group of $\mathfrak{g}$ realized as $\{g_{|\mathfrak{h}} | g \in G, g\mathfrak{h} = \mathfrak{h}\}$. In [HC2], Harish-Chandra proved that there exists an algebra homomorphism $\delta : \mathcal{D}(\mathfrak{g})^G \longrightarrow \mathcal{D}(\mathfrak{h})^W$ with

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the following properties. If f is a G-invariant polynomial on $\mathfrak{g}$ looked upon as a differential operator under multiplication then $\delta(f) = f_{|\mathfrak{h}}$. If D is a constant coefficient G-invariant differential operator on $\mathfrak{g}$ looked upon as an element of the symmetric algebra of $\mathfrak{g}$ (in turn identified with the polynomials on $\mathfrak{g}^*$) then $\delta(D)$ is the restriction to $\mathfrak{h}^*$ relative to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus [\mathfrak{h}, \mathfrak{g}]$. The kernel of δ is the space of $D \in \mathcal{D}(\mathfrak{g})^G$ such that $Df = 0$ for any G-invariant polynomial f on $\mathfrak{g}$. Harish-Chandra's proof of the existence of δ uses analysis to show that certain operators that *a priori* have rational coefficients actually have polynomial coefficients.

The main purpose of this paper is to give a completely algebraic construction of δ analogous to that of the Harish-Chandra isomorphism from the center of the enveloping algebra to the Weyl group invariants in the enveloping algebra of $\mathfrak{h}$. Our proof that our construction does indeed yield δ has two ingredients. The first is the recent theorem (see [LS] and [Wa1]) that $\mathcal{D}(\mathfrak{h})^W$ is generated by the W-invariant polynomials and the W-invariant constant coefficient differential operators on $\mathfrak{h}$. The second is the analysis of a certain module, M, for $\mathcal{D}(\mathfrak{g})$ analogous to a Verma module for $\mathfrak{g}$. To give an idea of the delicacy of our result we show how it implies Freudenthal's "strange formula" for the square of the norm of ρ relative to the Killing form on a semisimple Lie algebra. We actually give two proofs of the main result. The second involves applying the Zuckerman functors to M thereby constructing a $(\mathcal{D}(\mathfrak{g}), G)$-module. Under suitable hypotheses we prove that the module is holonomic. In Section 8 we give a conjectural decomposition of this module of the form

$$\bigoplus_{\gamma \in \hat{W}} (\dim \gamma) N^\gamma$$

where the N^γ are pairwise non-isomorphic irreducible $\mathcal{D}(\mathfrak{g})$-modules and $\hat{W}$ is the set of equivalence classes of irreducible finite-dimensional representations of W. We check this conjecture for $\mathfrak{g} = \mathfrak{sl}_2$.

It is our pleasure to dedicate this article to Ray Kunze in commemoration of his 65th birthday.

1. A homomorphism due to Harish-Chandra

Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$ with adjoint group G. The adjoint action of G on $\mathfrak{g}$ induces actions on the algebra of polynomial functions $\mathcal{O}(\mathfrak{g})$ and hence on the (non-commutative) algebra of polynomial differential operators $\mathcal{D}(\mathfrak{g}) \subset \mathrm{End}_{\mathbb{C}}(\mathcal{O}(\mathfrak{g}))$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, H the corresponding Cartan subgroup of G. Then the Weyl group $W = N_G(H)/H$ acts on $\mathfrak{h}$, hence on $\mathcal{O}(\mathfrak{h})$ and on $\mathcal{D}(\mathfrak{h})$. In [HC2], Harish-Chandra constructed an algebra homomorphism $\delta : \mathcal{D}(\mathfrak{g})^G \longrightarrow \mathcal{D}(\mathfrak{h})^W$ which extends the Chevalley restriction mappings $\mathcal{O}(\mathfrak{g})^G \longrightarrow \mathcal{O}(\mathfrak{h})^W$ and $S(\mathfrak{g})^G \longrightarrow S(\mathfrak{h})^W$.

We recall briefly the construction of δ. Let Φ be the root system of $(\mathfrak{g}, \mathfrak{h})$. Choose Φ_+, a system of positive roots. Set $\pi = \prod_{\alpha \in \Phi_+} \alpha$ and $\mathfrak{h}' = \{h \in$

$\mathfrak{h} \mid \pi(h) \neq 0\}$. Then $\mathfrak{h}'$ is a W-invariant affine open subset of $\mathfrak{h}$ on which W acts freely. Therefore, if $D \in \mathcal{D}(\mathfrak{g})^G$ then there exists a unique element $\delta'(D) \in \mathcal{D}(\mathfrak{h}')^W$ such that

$$\delta'(D)(f_{|\mathfrak{h}}) = D(f)_{|\mathfrak{h}} \qquad \text{for } f \in \mathcal{O}(\mathfrak{g})^G.$$

Now define $\delta(D) = \pi\delta'(D)\pi^{-1}$. A priori, δ is a homomorphism of $\mathcal{D}(\mathfrak{g})^G$ into $\mathcal{D}(\mathfrak{h}')^W$. Harish-Chandra's formula in [HC1] for the radial component of a reductive Lie algebra implies that δ extends the restriction mapping $S(\mathfrak{g})^G \to S(\mathfrak{h})^W$. In [HC2], Harish-Chandra proved that for each $D \in \mathcal{D}(\mathfrak{g})$, $\delta(D)$ is regular on $\mathfrak{h}$, i.e. $\delta(D) \in \mathcal{D}(\mathfrak{h})^W$.

The kernel of δ is the ideal $\mathcal{I} = \{D \in \mathcal{D}(\mathfrak{g})^G \mid D(f) = 0 \text{ for all } f \in \mathcal{O}(\mathfrak{g})^G\}$. Recently, Levasseur and Stafford [LS] proved that the following sequence is exact:

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{D}(\mathfrak{g})^G \xrightarrow{\delta} \mathcal{D}(\mathfrak{h})^W \to 0.$$

More precisely they showed that if $\mathcal{A}$ is the subalgebra of $\mathcal{D}(\mathfrak{g})^G$ generated by the subspaces $\mathcal{O}(\mathfrak{g})^G$ and $S(\mathfrak{g})^G$ then $\delta(\mathcal{A}) = \mathcal{D}(\mathfrak{h})^W$. This is a consequence of the following result (cf. [LS, Theorem 3]):

THEOREM 1.1. *Let V be a finite-dimensional vector space over $\mathbb{C}$ and let W be a finite group acting linearly on V. Then $\mathcal{D}(V)^W$ is generated (as an algebra) by the subspaces $\mathcal{O}(V)^W$ and $S(V)^W$.*

We will use this theorem and the construction of the next section to give an algebraic proof of the fact that $\delta(\mathcal{D}(\mathfrak{g})^G) \subset \mathcal{D}(\mathfrak{h})^W$.

2. An algebraic construction of δ

We retain the notation of the previous section. Again, we fix Φ_+, a system of positive roots for Φ. Set $\mathfrak{n}_+ = \sum_{\alpha\in\Phi_+} \mathfrak{g}_\alpha$ and $\mathfrak{n}_- = \sum_{\alpha\in\Phi_+} \mathfrak{g}_{-\alpha}$. Then we have a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. This decomposition gives also a decomposition of the dual $\mathfrak{g}^* = \mathfrak{n}_-^* \oplus \mathfrak{h}^* \oplus \mathfrak{n}_+^*$. We note that if V is a finite-dimensional vector space then $\mathcal{D}(V) = S(V^*)S(V)$ and also $\mathcal{D}(V) = S(V)S(V^*)$. Thus, we may write $\mathcal{D}(\mathfrak{g})$ in the following way:

$$\begin{aligned}\mathcal{D}(\mathfrak{g}) &= S(\mathfrak{g}^*)S(\mathfrak{g}) \\ &= S(\mathfrak{n}_-^*)S(\mathfrak{n}_-)\, S(\mathfrak{h}^*)S(\mathfrak{h})\, S(\mathfrak{n}_+^*)S(\mathfrak{n}_+) \\ &= S(\mathfrak{n}_-)S(\mathfrak{n}_-^*)\, S(\mathfrak{h}^*)S(\mathfrak{h})\, S(\mathfrak{n}_+^*)S(\mathfrak{n}_+).\end{aligned}$$

This provides a linear isomorphism

$$\mathcal{D}(\mathfrak{g}) = S(\mathfrak{n}_+^* \oplus \mathfrak{n}_-) \otimes \mathcal{D}(\mathfrak{h}) \otimes S(\mathfrak{n}_-^* \oplus \mathfrak{n}_+)$$

which induces a decomposition

$$\mathcal{D}(\mathfrak{g}) = \mathcal{D}(\mathfrak{h}) \oplus ((\mathfrak{n}_+^* \oplus \mathfrak{n}_-)\mathcal{D}(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})(\mathfrak{n}_-^* \oplus \mathfrak{n}_+)).$$

Let p denote the corresponding projection of $\mathcal{D}(\mathfrak{g})$ onto $\mathcal{D}(\mathfrak{h})$. Clearly, the map p extends the restriction mappings on $\mathcal{O}(\mathfrak{g})$ and on $S(\mathfrak{g})$. Our goal is to prove the following

THEOREM 2.1. *The restriction of p to $\mathcal{D}(\mathfrak{g})^G$ is equal to δ.*

REMARK. This proves in particular that $\delta(\mathcal{D}(\mathfrak{g})^G) \subset \mathcal{D}(\mathfrak{h})^W$.

The proof depends on three lemmas.

LEMMA 2.2. *The restriction of p to $\mathcal{D}(\mathfrak{g})^H$ is an algebra homomorphism into $\mathcal{D}(\mathfrak{h})$.*

PROOF. Let $\alpha_1, \dots, \alpha_r$ be the positive roots. Choose a basis $X_1, \dots, X_r$ of $\mathfrak{n}_+$ with $X_i \in \mathfrak{g}_{\alpha_i}$ and a basis $Y_1, \dots, Y_r$ of $\mathfrak{n}_-$ with $Y_i \in \mathfrak{g}_{-\alpha_i}$. Let $x_1, \dots, x_r$ and $y_1, \dots, y_r$ denote the corresponding linear coordinates on $\mathfrak{n}_+$ and $\mathfrak{n}_-$. If $I = (i_1, \dots, i_r) \in \mathbb{N}^r$ we define $x^I = x_1^{i_1} \cdots x_r^{i_r}, X^I = X_1^{i_1} \cdots X_r^{i_r}$ and similarly y^I and Y^I.

The elements $x^I Y^{J'} y^J X^{I'}$ $(I, J, I', J' \in \mathbb{N}^r)$, form a basis of $\mathcal{D}(\mathfrak{g})$ as a (left or right) $\mathcal{D}(\mathfrak{h})$-module. The submodule $\mathcal{D}(\mathfrak{g})^H$ is spanned by the elements of this basis with $\sum_k (i_k + j_k')\alpha_k = \sum_k (j_k + i_k')\alpha_k$. This clearly implies

$$\begin{aligned}\mathcal{D}(\mathfrak{g})^H \cap ((\mathfrak{n}_+^* \oplus \mathfrak{n}_-)\mathcal{D}(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})(\mathfrak{n}_-^* \oplus \mathfrak{n}_+)) &= \mathcal{D}(\mathfrak{g})^H \cap \mathcal{D}(\mathfrak{g})(\mathfrak{n}_-^* \oplus \mathfrak{n}_+) \\ &= \mathcal{D}(\mathfrak{g})^H \cap (\mathfrak{n}_+^* \oplus \mathfrak{n}_-)\mathcal{D}(\mathfrak{g}).\end{aligned}$$

Now, if $D_1, D_2 \in \mathcal{D}(\mathfrak{g})^H$ then

$$D_1 D_2 \equiv D_1 p(D_2) \pmod{\mathcal{D}(\mathfrak{g})(\mathfrak{n}_-^* \oplus \mathfrak{n}_+)}$$

and finally,

$$D_1 D_2 \equiv p(D_1)p(D_2) \pmod{(\mathfrak{n}_+^* \oplus \mathfrak{n}_-)\mathcal{D}(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})(\mathfrak{n}_-^* \oplus \mathfrak{n}_+)}.$$

This is the content of the lemma. □

LEMMA 2.3. *$p(\mathcal{D}(\mathfrak{g})^G)$ is contained in $\mathcal{D}(\mathfrak{h})^W$.*

PROOF. If $\alpha \in \Phi$ we denote by s_α the reflection in W associated to α. Let Σ denote the system of simple roots in Φ_+. The Weyl group W is generated by the s_α for $\alpha \in \Sigma$. Therefore, the Lemma follows if we show that $p(\mathcal{D}(\mathfrak{g})^G)$ is contained in the s_α-invariants in $\mathcal{D}(\mathfrak{h})$ for $\alpha \in \Sigma$.

Fix an element $\alpha \in \Sigma$. Let $\mathfrak{m} = \mathfrak{g}_{-\alpha} \oplus \mathfrak{h} \oplus \mathfrak{g}_\alpha$. Set $\mathfrak{u}_+ = \sum_{\beta \in \Phi_+ \setminus \{\alpha\}} \mathfrak{g}_\beta$ and $\mathfrak{u}_- = \sum_{\beta \in \Phi_+ \setminus \{\alpha\}} \mathfrak{g}_{-\beta}$. Then $\mathfrak{g} = \mathfrak{u}_- \oplus \mathfrak{m} \oplus \mathfrak{u}_+$ and we obtain a direct sum decomposition

$$\mathcal{D}(\mathfrak{g}) = \mathcal{D}(\mathfrak{m}) \oplus ((\mathfrak{u}_+^* \oplus \mathfrak{u}_-)\mathcal{D}(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})(\mathfrak{u}_-^* \oplus \mathfrak{u}_+)).$$

Let q be the corresponding projection of $\mathcal{D}(\mathfrak{g})$ onto $\mathcal{D}(\mathfrak{m})$. Let M be the adjoint group of the reductive Lie algebra $\mathfrak{m}$. We have $[\mathfrak{m}, \mathfrak{u}_+] \subset \mathfrak{u}_+$ and $[\mathfrak{m}, \mathfrak{u}_-] \subset \mathfrak{u}_-$. This implies that $q(\mathcal{D}(\mathfrak{g})^M)$ is contained in $\mathcal{D}(\mathfrak{m})^M$. Since $p \circ q = p$, we may assume that $\mathfrak{g} = \mathfrak{m}$, i.e. $\Phi_+ = \{\alpha\}$.

We are reduced to the case when $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{sl}_2$. Let X_1, X_2, X_3 be an orthonormal basis of $\mathfrak{g}$ with respect to the Killing form. Let x_1, x_2, x_3 denote the corresponding linear coordinates. Using classical invariant theory we see that $\mathcal{D}(\mathfrak{g})^G$ is generated as an algebra by the Laplacian $\Delta = X_1^2 + X_2^2 + X_3^2$, the quadratic form $r^2 = x_1^2 + x_2^2 + x_3^2$ and the Euler operator $E = x_1X_1 + x_2X_2 + x_3X_3$. Set $e = \Delta$, $f = -\frac{1}{4}r^2$ and $h = -E - \frac{3}{2}$. Then h, e, f satisfy the relations $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. Thus, h, e, f span a three dimensional simple Lie algebra $\mathfrak{s}$ and we obtain an isomorphism of algebras $\mathcal{D}(\mathfrak{g})^G = U(\mathfrak{s})$. (Here $U(\mathfrak{s})$ denotes the universal enveloping algebra of $\mathfrak{s}$.) In particular, $\mathcal{D}(\mathfrak{g})^G$ is generated as an algebra by $\mathcal{O}(\mathfrak{g})^G = \mathbb{C}[r^2]$ and $S(\mathfrak{g})^G = \mathbb{C}[\Delta]$ in this case. Hence $p(\mathcal{D}(\mathfrak{g})^G)$ is contained in $\mathcal{D}(\mathfrak{h})^W$. □

So far we have proved that p is an algebra homomorphism of $\mathcal{D}(\mathfrak{g})^G$ into $\mathcal{D}(\mathfrak{h})^W$ which is equal to δ on the subalgebra $\mathcal{A}$ of $\mathcal{D}(g)^G$ generated by $\mathcal{O}(\mathfrak{g})^G$ and $S(\mathfrak{g})^G$. Let $\mathcal{J}$ be the kernel of the homomorphism $p : \mathcal{D}(\mathfrak{g})^G \longrightarrow \mathcal{D}(\mathfrak{h})^W$. In light of Theorem 1.1 we have $\mathcal{D}(\mathfrak{g})^G = \mathcal{J} + \mathcal{A}$. Therefore, if we show that $\mathcal{I} = \mathcal{J}$ then Theorem 2.1 follows.

Recall that the differential of the action of G on $\mathfrak{g}$ provides a Lie algebra homomorphism $\tau : \mathfrak{g} \hookrightarrow \mathrm{Der}(\mathcal{O}(\mathfrak{g})) \subset \mathcal{D}(\mathfrak{g})$. Clearly, $\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) \cap \mathcal{D}(\mathfrak{g})^G \subset \mathcal{I}$.

LEMMA 2.4. $\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) \cap \mathcal{D}(\mathfrak{g})^G \subset \mathcal{J}$.

Assuming the assertion of Lemma 2.4 we now complete the proof of Theorem 2.1.

PROOF OF THEOREM 2.1. Let d be the discriminant of $\mathfrak{g}$, i.e. d is the non-zero polynomial on $\mathfrak{g}$ such that

$$\det(t - \mathrm{ad}(X)) = d(X)t^l + \text{ terms of higher order in } t$$

for all $X \in \mathfrak{g}$ (with t an indeterminate). Then $d \in \mathcal{O}(\mathfrak{g})^G$ and $d_{|\mathfrak{h}} = \pi^2$. In [Wa1] we proved that $(D(\mathfrak{g})\tau(\mathfrak{g}) \cap \mathcal{D}(\mathfrak{g})^G)_{[d]} = \mathcal{I}_{[d]}$. Here the subscript $[d]$ means localization at d. Lemma 2.4 implies $\mathcal{I}_{[d]} \subset \mathcal{J}_{[d]}$. Now we prove that the inclusion is in fact an equality. We note that $\mathcal{D}(\mathfrak{h}')^W = \mathcal{D}(\mathfrak{h})^W_{[\pi^2]}$ and that the homomorphism $\delta' : \mathcal{D}(\mathfrak{g})^G \longrightarrow \mathcal{D}(\mathfrak{h})^W_{[\pi^2]}$ has a natural extension to a homomorphism $\delta' : \mathcal{D}(\mathfrak{g})^G_{[d]} \longrightarrow \mathcal{D}(\mathfrak{h})^W_{[\pi^2]}$. Theorem 1.1 together with Harish-Chandra's formula for δ' on $\mathcal{A}$ implies that the following sequence is exact:

$$0 \longrightarrow \mathcal{I}_{[d]} \longrightarrow \mathcal{D}(\mathfrak{g})^G_{[d]} \xrightarrow{\delta'} \mathcal{D}(\mathfrak{h})^W_{[\pi^2]} \longrightarrow 0.$$

We observe that $\mathcal{D}(\mathfrak{h})^W_{[\pi^2]}$ is a simple algebra. This follows easily from the fact that the algebra $\mathcal{D}(\mathfrak{h})^W$ is simple (cf. [Wa1, Lemma 1.2]). Thus, $\mathcal{I}_{[d]}$ is a maximal ideal of $\mathcal{D}(\mathfrak{g})^G_{[d]}$. Hence $\mathcal{I}_{[d]} = \mathcal{J}_{[d]}$. This implies $\mathcal{I} = \mathcal{J}$. □

We will give two proofs of Lemma 2.4 in sections 4 and 5. In the next section we want to show that Lemma 2.4 surprisingly implies Freudenthal's "strange formula."

3. Freudenthal's "strange formula"

In this section we assume that $\mathfrak{g}$ is semisimple. Let B denote the Killing form on $\mathfrak{g}$. If $\lambda \in \mathfrak{h}^*$ then define $H_\lambda \in \mathfrak{h}$ by $B(H_\lambda, h) = \lambda(h)$ for $h \in \mathfrak{h}$. Let $(\ ,\ )$ be the bilinear form on $\mathfrak{h}^*$ given by $(\lambda, \mu) = B(H_\lambda, H_\mu)$ for $\lambda, \mu \in \mathfrak{h}^*$. Freudenthal proved the following formula, which is now known as Freudenthal's "strange formula":

THEOREM 3.1. *Let* $\rho = \frac{1}{2}\sum_{\alpha \in \Phi_+} \alpha$. *Then*

$$\dim \mathfrak{g} = \frac{1}{24}(\rho, \rho).$$

We want to show that this formula is implied by Lemma 2.4.

LEMMA 3.2. *Let* $\Omega \in U(\mathfrak{g})$ *be the Casimir element in the universal enveloping algebra of* $\mathfrak{g}$. *Then*

$$p(\tau(\Omega)) = 6(\rho, \rho) - \frac{1}{4}\sum_{\alpha \in \Phi}(\alpha, \alpha) - \frac{1}{4}|\Phi|.$$

PROOF. Let $X_1, \ldots, X_n$ be an orthonormal basis of $\mathfrak{g}$ (with respect to B). Then $\Omega = \sum_k X_k^2$. Hence

$$\begin{aligned}
\tau(\Omega) &= \sum_k \tau(X_k)^2 \\
&= \sum_{i,j,k} X_i^*[X_k, X_i]X_j^*[X_k, X_j] \\
&= \sum_{i,j,k} X_i^* B([X_k, X_i], X_j)[X_k, X_j] + \sum_{i,j,k} X_i^* X_j^*[X_k, X_i][X_k, X_j] \\
&= \sum_i X_i^* X_i + \sum_{i,j,k} X_i^* X_j^*[X_k, X_i][X_k, X_j] \\
&= E + D
\end{aligned}$$

with E the Euler operator on $\mathfrak{g}$. In order to calculate $p(D)$ we choose another basis of $\mathfrak{g}$: For $\alpha \in \Phi_+$ let $X_\alpha \in \mathfrak{g}_\alpha$ and $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $B(X_\alpha, X_{-\alpha}) = 1$. Furthermore let $H_1, \ldots, H_l$ be an orthonormal basis of $\mathfrak{h}$. Then, modulo

$(\mathfrak{n}_+^* \oplus \mathfrak{n}_-)\mathcal{D}(\mathfrak{g}) + \mathcal{D}(\mathfrak{g})(\mathfrak{n}_-^* \oplus \mathfrak{n}_+)$, we obtain

$$\begin{aligned}
D &\equiv \sum_{\alpha,\beta,k} X_\alpha^* X_\beta^* [X_k, X_\alpha][X_k, X_\beta] + 2\sum_{\beta,i,k} H_i^* X_\beta^* [X_k, H_i][X_k, X_\beta] \\
&\quad + \sum_{i,j,k} H_i^* H_j^* [X_k, H_i][X_k, H_j] \\
&\equiv \sum_{\alpha,\beta\prec 0,k} X_\alpha^* X_\beta^* [X_k, X_\alpha][X_k, X_\beta] + 2\sum_{\beta\prec 0,i,k} H_i^* X_\beta^* [X_k, H_i][X_k, X_\beta] \\
&\equiv \sum_{\alpha,\beta\prec 0,k} B([X_k, X_\alpha], X_{-\beta}) B([X_k, X_\beta], X_{-\beta}) \\
&\quad + \sum_{\alpha,\beta\prec 0,k} B([X_k, X_\alpha], X_{-\alpha}) B([X_k, X_\beta], X_{-\beta}) \\
&\quad - 2\sum_{\beta\prec 0,i,k} H_i^* \beta(H_i) B(X_k, X_{-\beta})[X_k, X_\beta] \\
&\equiv \sum_{\alpha,\beta\prec 0} B([X_\beta, [X_{-\beta}, X_\alpha]], X_{-\alpha}) + \sum_{\alpha,\beta\prec 0} B(H_\alpha, H_\beta) + 2\sum_{\beta\prec 0} \beta H_\beta \\
&\equiv C_1 + C_2 + E'
\end{aligned}$$

with C_1, C_2 constants and E' the Euler operator on $\mathfrak{h}$. Clearly, $C_2 = 4(\rho, \rho)$. It remains to determine C_1. We note that in $U(\mathfrak{g})$ we have the identity

$$\sum_{\beta\prec 0} X_\beta X_{-\beta} = \frac{1}{2}\Omega - H_\rho - \frac{1}{2}\sum_i H_i^2.$$

Hence

$$\sum_{\beta\prec 0} [X_\beta, [X_{-\beta}, X_\alpha]] = \frac{1}{2} X_\alpha - (\rho, \alpha) X_\alpha - \frac{1}{2}(\alpha, \alpha) X_\alpha.$$

This now implies

$$C_1 = \frac{1}{2}|\Phi_+| + 2(\rho, \rho) - \frac{1}{2}\sum_{\alpha\prec 0} (\alpha, \alpha).$$

Finally, we obtain

$$\begin{aligned}
p(\tau(\Omega)) &= p(E) + p(D) \\
&= E' - |\Phi_+| + C_1 + C_2 - E' \\
&= 6(\rho, \rho) - \frac{1}{2}\sum_{\alpha\in\Phi_+} (\alpha, \alpha) - \frac{1}{2}|\Phi_+|.
\end{aligned}$$

This completes the proof. □

On the other hand, Lemma 2.4 implies $p(\tau(\Omega)) = 0$. Thus, we have proved the formula

$$24(\rho, \rho) = \sum_{\alpha\in\Phi} (\alpha, \alpha) + |\Phi|.$$

Freudenthal's formula is now a consequence of the following nice identity (cf. [Br]):

LEMMA 3.3.

$$\dim \mathfrak{h} = \sum_{\alpha \in \Phi} (\alpha, \alpha).$$

PROOF. The dimension of $\mathfrak{h}$ is the rank of the Gram matrix $A = [(\alpha, \beta)]_{\alpha,\beta \in \Phi}$. We now observe that

$$(\alpha, \beta) = \operatorname{tr} \operatorname{ad}(H_\alpha) \operatorname{ad}(H_\beta) = \sum_{\gamma \in \Phi} \gamma(H_\alpha)\gamma(H_\beta) = \sum_{\gamma \in \Phi} (\alpha, \gamma)(\gamma, \beta).$$

Thus, the matrix A (looked upon as a linear operator on $\mathbb{C}^{|\Phi|}$) is idempotent. Hence the rank of A is equal to the trace of A. This implies the asserted identity. □

4. A $(\mathfrak{g}, \mathcal{D}(\mathfrak{g})^G)$-bimodule and the proof of the main lemma

The key to the proof of Lemma 2.4 is a certain $\mathcal{D}(\mathfrak{g})$-module, which we now introduce. Let $S(\mathfrak{h} \oplus \mathfrak{n}_-^* \oplus \mathfrak{n}_+)$ act on $\mathbb{C}$ by evaluation at 0. (Here we identify $S(\mathfrak{h} \oplus \mathfrak{n}_-^* \oplus \mathfrak{n}_+)$ with the polynomials on $\mathfrak{h}^* \times \mathfrak{n}_- \times \mathfrak{n}_+^*$.) Then we define M to be the $\mathcal{D}(\mathfrak{g})$-(left)module

$$M = \mathcal{D}(\mathfrak{g}) \otimes_{S(\mathfrak{h} \oplus \mathfrak{n}_-^* \oplus \mathfrak{n}_+)} \mathbb{C}.$$

We also look upon M as a $\mathfrak{g}$-module via τ. Note that this $\mathfrak{g}$-action commutes with the action of the invariant differential operators $\mathcal{D}(\mathfrak{g})^G$.

The weight structure of the $\mathfrak{g}$-module M will be essential.

LEMMA 4.1. *Let $\alpha_1, \dots, \alpha_r$ be the (distinct) elements of Φ_+. Let $\rho = \frac{1}{2} \sum_i \alpha_i$. Choose a basis $X_1, \dots, X_r$ of $\mathfrak{n}_+$ with $X_i \in \mathfrak{g}_{\alpha_i}$ and a basis $Y_1, \dots, Y_r$ of $\mathfrak{n}_-$ with $Y_i \in \mathfrak{g}_{-\alpha_i}$. Let $x_1, \dots, x_r$ and $y_1, \dots, y_r$ denote the corresponding linear coordinates on $\mathfrak{n}_+$ and $\mathfrak{n}_-$. Then*

i) *The $\mathfrak{g}$-module M is $\mathfrak{h}$-diagonalizable, i.e. is M admits a decompostion $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ into weight spaces M_λ.*

ii) *If $\lambda \in \mathfrak{h}^*$ then the weight space M_λ is given as*

$$M_\lambda = \bigoplus_{\substack{I, J \in \mathbb{N}^r \\ -2\rho - \sum_k (i_k + j_k)\alpha_k = \lambda}} \mathcal{O}(\mathfrak{h})\, x^I Y^J \otimes 1.$$

PROOF. The elements $x^I Y^J \otimes 1 \quad (I, J \in \mathbb{N}^r)$, form a basis of M as a $\mathcal{O}(\mathfrak{h})$-module. For $h \in \mathfrak{h}$ we calculate the action of $\tau(h)$ on an element of this basis. First we consider the special case when $I = J = 0$. We note that $\tau(h) = -\sum_k \alpha_k(h)(x_k X_k - y_k Y_k)$. Thus,

$$\begin{aligned} \tau(h)\, 1 \otimes 1 &= \sum_k \alpha_k(h) y_k Y_k \otimes 1 \\ &= \sum_k \alpha_k(h)(Y_k y_k - 1) \otimes 1 \\ &= -2\rho(h) \otimes 1. \end{aligned}$$

In the general case we write

$$\tau(h)\, x^I Y^J \otimes 1 = x^I Y^J \tau(h) \otimes 1 + [\tau(h), x^I Y^J] \otimes 1.$$

A straightforward verification shows that

$$[\tau(h), x^I Y^J] \otimes 1 = -\sum_k (i_k + j_k)\alpha_k(h) x^I Y^J \otimes 1.$$

Hence we see that $x^I Y^J \otimes 1$ is a weight vector of weight $-2\rho - \sum_k (i_k + j_k)\alpha_k$. The claims of the lemma now follow. □

The homomorphism $p : \mathcal{D}(\mathfrak{g})^G \longrightarrow \mathcal{D}(\mathfrak{h})^W$ enters the picture when we look at the action of $\mathcal{D}(\mathfrak{g})^G$ on the -2ρ-weight space $M_{-2\rho} = \mathcal{O}(\mathfrak{h}) \otimes 1$.

LEMMA 4.2 If $D \in \mathcal{D}(\mathfrak{g})^G$ and $f \in \mathcal{O}(\mathfrak{h})$ then $D f \otimes 1 = (p(D)(f)) \otimes 1$.

PROOF. If $D \in \mathcal{D}(\mathfrak{g})^G$ then $D = p(D) + D'$ with $D' \in \mathcal{D}(\mathfrak{g})(\mathfrak{n}_- \oplus \mathfrak{n}_+)$. Hence $D f \otimes 1 = p(D) f \otimes 1$. We note that $\mathcal{D}(\mathfrak{h})\, 1 \otimes 1 = \mathcal{D}(\mathfrak{h}) \otimes_{S(\mathfrak{h})} \mathbb{C} = \mathcal{O}(\mathfrak{h})$ with the usual action. This means that $p(D) f \otimes 1 = (p(D)(f)) \otimes 1$. □

The two preceeding lemmas provide all the information on M that we need to prove Lemma 2.4.

PROOF OF LEMMA 2.4. Let B be a non-degenerate invariant bilinear form on $\mathfrak{g}$. We assume that B restricts to the Killing form on $[\mathfrak{g}, \mathfrak{g}]$. Let $X_1, \ldots, X_n$ be an orthonormal basis of $\mathfrak{g}$ with respect to B.

Let $D \in \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}) \cap \mathcal{D}(\mathfrak{g})^G$. Write $D = \sum_i D_i \tau(X_i)$. Let ψ be the linear map from $\mathfrak{g}$ to $\mathcal{D}(\mathfrak{g})$ defined by sending X_i to D_i.

CLAIM. We may assume that $\psi \in \mathrm{Hom}_G(\mathfrak{g}, \mathcal{D}(\mathfrak{g}))$, i.e. $g\psi(X)g^{-1} = \psi(\mathrm{Ad}(g)X)$ for $g \in G, X \in \mathfrak{g}$.

PROOF OF THE CLAIM. If $g \in G$ let $\mathrm{Ad}(g)X_i = \sum_j a_{ji}(g) X_j$. Since $D \in \mathcal{D}(\mathfrak{g})^G$ we have $D = \sum_{i,j} a_{ji}(g) g D_i g^{-1} \tau(X_i)$. Let K be a maximal compact subgroup of G and let dk be a left invariant normalized measure on K. Define

$$\check{D}_i = \int_K dk \{\sum_j a_{ij}(k) k D_j k^{-1}\}.$$

(Those that prefer algebra can use an appropriate Reynolds operator in place of the integration.) Then $D = \sum_i \check{D}_i \tau(X_i)$ and $g\check{D}_i g^{-1} = \sum_j a_{ij}(g^{-1})\check{D}_j = \sum_j a_{ji}(g)\check{D}_j$. The claim follows. □

Choose for each $\alpha \in \Phi$ an element $X_\alpha \in \mathfrak{g}_\alpha$ such that $B(X_\alpha, X_{-\alpha}) = 1$. Let $H_1, \ldots, H_l$ be an orthonormal basis of $\mathfrak{h}$ with respect to B. Then

$$\begin{aligned} D &= \sum_{i,\alpha} B(X_\alpha, X_i) D_i \tau(X_\alpha) + \sum_{i,j} B(H_j, X_i) D_i \tau(H_j) \\ &= \sum_\alpha \psi(X_{-\alpha})\tau(X_\alpha) + \sum_j \psi(H_j)\tau(H_j). \end{aligned}$$

For $f \in \mathcal{O}(\mathfrak{h})$ we calculate

$$Df \otimes 1 = \sum_{\alpha \succ 0} \psi(X_\alpha)\tau(X_{-\alpha})f \otimes 1 - 2\sum_j \rho(H_j)\psi(H_j)f \otimes 1.$$

By the equivariance of ψ we have $\psi(X_\alpha)f \otimes 1 \in M_{-2\rho+\alpha}$. Lemma 4.1 implies $M_{-2\rho+\alpha} = 0$ for $\alpha \succ 0$. Hence

$$\psi(X_\alpha)\tau(X_{-\alpha})f \otimes 1 = [\psi(X_\alpha), \tau(X_{-\alpha})]f \otimes 1 = \psi(H_\alpha)f \otimes 1.$$

Thus, we obtain

$$\begin{aligned} Df \otimes 1 &= \sum_{i,\alpha \succ 0} \alpha(H_i)\psi(H_i)f \otimes 1 - 2\sum_j \rho(H_j)\psi(H_j)f \otimes 1 \\ &= 0. \end{aligned}$$

Lemma 4.2 now implies $p(D) = 0$. □

5. Applying Zuckerman functors to M

In the sequel, let $r = \frac{1}{2}\dim(\mathfrak{g}/\mathfrak{h})$. In this section we give a second proof of Lemma 2.4 which involves applying the Zuckerman functor $(\Gamma_H^G)^r = \Gamma^r$ to the module M. We start with the following observation.

LEMMA 5.1. *$U(\mathfrak{g})\mathcal{O}(\mathfrak{h}) \otimes 1$ is a $(\mathfrak{g}, \mathcal{D}(\mathfrak{g})^G)$-bimodule summand of M isomorphic to $M(-2\rho) \otimes \mathcal{O}(\mathfrak{h})$ with $\mathfrak{g}$ acting on the Verma module $M(-2\rho)$ with highest weight -2ρ and $\mathcal{D}(\mathfrak{g})^G$ acting on $\mathcal{O}(\mathfrak{h})$ via p.*

PROOF. If $f \in \mathcal{O}(\mathfrak{h})$, $f \neq 0$, then $\tau(U(\mathfrak{g}))f \otimes 1$ is a highest weight module with highest weight -2ρ. We note that every highest weight module with highest weight -2ρ is irreducible. This implies that $\tau(U(\mathfrak{g}))\mathcal{O}(\mathfrak{h}) \otimes 1 = M(-2\rho) \otimes \mathcal{O}(\mathfrak{h})$. (The action of $\mathcal{D}(\mathfrak{g})^G$ is the predicted one since the $\mathcal{D}(\mathfrak{g})^G$-action commutes with the $\mathfrak{g}$-action.) Next we observe that $\mathrm{Hom}_{\mathfrak{g}}(M(-2\rho), M) \otimes M(-2\rho)$ is a summand of M. Since $\mathcal{O}(\mathfrak{h}) \otimes 1$ is the full -2ρ-weight space we obtain the result. □

OUTLINE OF A PROOF OF LEMMA 2.4. Let s be the (unique) element of the Weyl group W such that $s(\Phi_+) = -\Phi_+$. Then $s(-2\rho + \rho) + \rho = 0$. This implies that if we apply the Zuckerman functor $(\Gamma_H^G)^r = \Gamma^r$ to the irreducible Verma module $M(-2\rho)$ then we obtain the trivial representation $\mathbb{C}$ (cf. [RRG, 6.6]). The idea is now to apply Γ^r to the module M (looked upon as a $(\mathfrak{g}, H)$-module). We will show that $\Gamma^r(M)$ admits a natural $\mathcal{D}(\mathfrak{g})$-module structure such that the $\mathfrak{g}$-action is the one induced by τ. Then $\Gamma^r(M)$ contains a $(\mathfrak{g}, \mathcal{D}(\mathfrak{g})^G)$-bimodule summand isomorphic to $\mathbb{C} \otimes \mathcal{O}(\mathfrak{h})$ with $\mathfrak{g}$ acting trivially on $\mathbb{C}$ and $\mathcal{D}(\mathfrak{g})^G$ acting on $\mathcal{O}(\mathfrak{h})$ via p. The assertion of Lemma 2.4 is now reduced to a triviality. If $D \in \mathcal{D}(\mathfrak{g})^G \cap \mathcal{D}(\mathfrak{g})\tau(\mathfrak{g})$ then we write $D = \sum_i D_i\tau(X_i)$ with $D_i \in \mathcal{D}(\mathfrak{g}), X_i \in \mathfrak{g}$. Applying D to $1 \otimes f \in \mathbb{C} \otimes \mathcal{O}(\mathfrak{h}) \subset \Gamma^r(M)$ we obtain $1 \otimes (p(D)(f)) = \sum_i D_i(X_i 1 \otimes f) = 0$. □

We now give the details needed in the proof above. We start by recalling the construction of the Zuckerman functors (cf. [Wa2]). For the moment let $G \supset H$

be arbitrary reductive linear algebraic groups over $\mathbb{C}$ with Lie algebras $\mathfrak{g} \supset \mathfrak{h}$. Assume that G is given as a closed subgroup of a linear algebraic group $\hat{G}$. Let $\hat{\mathfrak{g}}$ denote the Lie algebra of $\hat{G}$. A module M for $\hat{\mathfrak{g}}$ and H is called a $(\hat{\mathfrak{g}}, H)$-module if the following three conditions are satisfied:

i) If $X \in \hat{\mathfrak{g}}, h \in H$ then $hXm = \mathrm{Ad}(h)Xhm$ for all $m \in M$;

ii) As an H-module, M is locally finite, i.e. M is the union of finite-dimensional algebraic H-modules;

iii) If $V \subset M$ is a finite-dimensional H-submodule then V is $\mathfrak{h}$-invariant and the action of $\mathfrak{h}$ on V is the action induced by the differential of the action of H.

Morphisms of $(\hat{\mathfrak{g}}, H)$-modules are linear maps that are both $\hat{\mathfrak{g}}$- and H-module homomorphisms.

Let M be a $(\hat{\mathfrak{g}}, H)$-module. Let $\mathcal{O}(G)$ denote the ring of regular functions on G. If $X \in \hat{\mathfrak{g}}$ then we define an endomorphism $\mu(X)$ of $M \otimes \mathcal{O}(G)$ as follows. First we identify $M \otimes \mathcal{O}(G)$ with the space of mappings $\varphi : G \longrightarrow M$ such that $\varphi(G)$ is contained in a finite-dimensional subspace V of M and $\varphi : G \longrightarrow V$ is a morphism of algebraic varieties. (If $m \in M, f \in \mathcal{O}(G)$ then we set $(m \otimes f)(g) = f(g)m$ for $g \in G$.) Then we define

$$\mu(X)\varphi(g) = \mathrm{Ad}(g)X\varphi(g) \qquad \text{for } \varphi \in M \otimes \mathcal{O}(G), g \in G.$$

Clearly, if $X, Y \in \hat{\mathfrak{g}}$ then $\mu([X, Y]) = [\mu(X), \mu(Y)]$.

Let π denote the $(\hat{\mathfrak{g}}, H)$-action on M. On $\mathcal{O}(G)$ we have two natural $(\mathfrak{g}, G)$-module structures: Set $\rho_l(g)f(x) = f(g^{-1}x)$, $\rho_r(g)f(xg)$ for $f \in \mathcal{O}(G), g, x \in G$. We look upon $M \otimes \mathcal{O}(G)$ as a $(\mathfrak{g}, H)$-module under $\pi \otimes \rho_l$. One can check that for $X \in \hat{\mathfrak{g}}$, the endomorphism $\mu(X)$ of $M \otimes \mathcal{O}(G)$ is a $(\mathfrak{g}, H)$-homomorphism. Furthermore, for each $g \in G$, $I \otimes \rho_r(g)$ is a $(\mathfrak{g}, H)$-homomorphism of $M \otimes \mathcal{O}(G)$ to itself.

We now consider the relative Lie algebra cohomology of $(\mathfrak{g}, H)$ with coefficients in $M \otimes \mathcal{O}(G)$. Recall that the complex is given as

$$C^j(\mathfrak{g}, H; M \otimes \mathcal{O}(G)) = \mathrm{Hom}_H(\wedge^j(\mathfrak{g}/\mathfrak{h}), M \otimes \mathcal{O}(G))$$

with the usual coboundary operator of relative Lie algebra cohomology (cf. [BW]). We define

$$\Gamma^j(M) = H^j(\mathfrak{g}, H; M \otimes \mathcal{O}(G)).$$

For each $X \in \hat{\mathfrak{g}}$ the endomorphism $\mu(X)$ induces an endomorphism $\bar{\mu}(X)$ on $\Gamma^j(M)$. This defines a $\hat{\mathfrak{g}}$-module structure on $\Gamma^j(M)$. The G-action $I \otimes \rho_r$ on $M \otimes \mathcal{O}(\mathfrak{g})$ induces a G-module structure on $\Gamma^j(M)$. With respect to these two actions $\Gamma^j(M)$ is a $(\hat{\mathfrak{g}}, G)$-module (cf. [RRG, 6.3]).

Next we introduce a certain category of $\mathcal{D}$-modules. Let V be a finite-dimensional vector space over $\mathbb{C}$. Let H be a closed reductive subgroup of $\mathrm{GL}(V)$. A module M for $\mathcal{D}(V)$ and H is called a $(\mathcal{D}(V), H)$-module if the following two conditions hold:

i) If $D \in \mathcal{D}(V)$, $h \in H$ then $hDm = (hDh^{-1})hm$ for all $m \in M$;

ii) As an H-module, M is locally finite.

Let $\mathfrak{h}$ be the Lie algebra of H. A $(\mathcal{D}(V), H)$-module M is called compatible, if the induced $\mathfrak{h}$-action is given via $\tau : \mathfrak{h} \hookrightarrow \mathcal{D}(V)$ and if M is a $(\mathfrak{h}, H)$-module under this action.

LEMMA 5.2. *Let $\mathfrak{g} \supset \mathfrak{h}$ be reductive Lie algebras over $\mathbb{C}$ with adjoint groups $G \supset H$. Let M be a compatible $(\mathcal{D}(\mathfrak{g}), H)$-module. Then $\Gamma^j(M)$ admits a natural structure of a compatible $(\mathcal{D}(\mathfrak{g}), G)$-module.*

PROOF. We consider the Heisenberg algebra $\mathfrak{a} = \mathfrak{g} \oplus \mathfrak{g}^* \oplus \mathbb{C}1 \subset \mathcal{D}(\mathfrak{g})$. Note that $\mathfrak{a}$ is normalized by the action of G. Thus, me may look upon M as a $(\mathfrak{g} \ltimes \mathfrak{a}, H)$-module. ($G \subset G \ltimes A$ with A the Heisenberg group corresponding to $\mathfrak{a}$.) We obtain a $\mathfrak{g} \ltimes \mathfrak{a}$-action on $\Gamma^j(M)$. In particular $\Gamma^j(M)$ is a $U(\mathfrak{a})$-module. Let c denote the central element $1 \in \mathfrak{a}$. There is a canonical isomorphism of algebras $\mathcal{D}(\mathfrak{g}) = U(\mathfrak{a})/U(\mathfrak{a})(c-1)$. The $U(\mathfrak{a})$-action on $\Gamma^j(M)$ induces a $\mathcal{D}(\mathfrak{g})$-action if $\mu(c) = 1$ on $M \otimes \mathcal{O}(G)$. But this is clear since $\mathrm{Ad}(g)c = c$ for $g \in G$ and c acts trivially on M.

We now show that the $\mathfrak{g}$-action on $\Gamma^j(M)$ is equal to the action of $\tau(\mathfrak{g}) \subset \mathcal{D}(\mathfrak{g})$. Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$. If $X \in \mathfrak{g}$ then $\tau(X) = -\sum_i X_i^*[X, X_i] \in \mathcal{D}(\mathfrak{g})$. We calculate the action of $-\sum_i X_i^*[X, X_i] \in U(\mathfrak{a})$ on $M \otimes \mathcal{O}(G)$:

$$\begin{aligned}\mu(-\sum_i X_i^*[X, X_i])\varphi(g) &= -\sum_i \mathrm{Ad}(g)X_i^* \,\mathrm{Ad}(g)[X, X_i]\varphi(g) \\ &= -\sum_i X_i^*[\mathrm{Ad}(g)X, X_i]\varphi(g) \\ &= \tau(\mathrm{Ad}(g)X)\varphi(g)\end{aligned}$$

for $\varphi \in M \otimes \mathcal{O}(G), g \in G$. Hence the result. □

We return to our previous situation. That is, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and M is the module defined in Section 4. Then M is a compatible $(\mathcal{D}(\mathfrak{g}), H)$-module and we obtain $(\mathcal{D}(\mathfrak{g}), G)$-modules $\Gamma^j(M)$ with the properties needed in the outline of the proof of Lemma 2.4 above. In the next sections we continue the study of these modules.

6. The case of $\mathfrak{g} = \mathfrak{sl}_2$

If $\mathfrak{g} = \mathfrak{sl}_2$ then it is possible to describe M very explicitly. The key is the analysis of the formal character of M, which we will introduce for general $\mathfrak{g}$.

We decompose the $(\mathfrak{g}, H)$-module M into modules from the category $\mathcal{O}$. Let $E \in \mathcal{D}(\mathfrak{g})$ be the Euler operator. Then E acts semisimply with integer eigenvalues on M. If $i \in \mathbb{Z}$ then we define

$$M^i = \{m \in M \mid Em = (i - r)m\}.$$

Since E commutes with the adjoint vector fields $\tau(\mathfrak{g})$ we have a decomposition $M = \bigoplus_i M^i$ into $\mathfrak{g}$-submodules. These modules are from the category $\mathcal{O}$. In

particular, their weight spaces are finite-dimensional. Recall that the formal character of M^i is given as

$$\operatorname{ch} M^i = \sum_{\lambda \in \mathfrak{h}^*} (\dim M^i_\lambda) e^\lambda.$$

We also consider a graded formal character for the graded object M. Set

$$\operatorname{ch}(M, q) = \sum_i \sum_{\lambda \in \mathfrak{h}^*} (\dim M^i_\lambda) e^\lambda q^i$$

with q an indeterminate. Using Lemma 4.1 we obtain a presentation of $\operatorname{ch}(M, q)$ as a rational function

$$\operatorname{ch}(M, q) = \frac{e^{-2\rho}}{(1-q)^l \prod_{\alpha \in \Phi_+} (1 - qe^{-\alpha})(1 - q^{-1}e^{-\alpha})}.$$

Let us now compute this character for $\mathfrak{g} = \mathfrak{sl}_2$. Set $x = e^{-\alpha}$. The key is to expand

$$\begin{aligned} \frac{1-x}{(1-qx)(1-q^{-1}x)} &= \sum_{k \geq 0} x^k (q^{-k} - q^{-k+1} + q^{-k+2} - \cdots + q^k) \\ &= \sum_{k \geq 0} x^k q^{-k} (1 - q + q^2 - \cdots + q^{2k}) \\ &= \sum_{k \geq 0} x^k q^{-k} \frac{1 + q^{2k+1}}{1+q} \end{aligned}$$

Thus,

$$\operatorname{ch}(M, q) = \frac{x}{(1-q)(1-qx)(1-qx^{-1})} = \sum_{k \geq 0} \frac{x^{k+1}}{1-x} \cdot \left(\frac{q^{-k}}{1-q^2} + \frac{q^{k+1}}{1-q^2}\right).$$

From this formula we can read the explicit structure of M as a $(\mathfrak{g}, \mathcal{D}(\mathfrak{g})^G)$-bimodule. Recall that $\mathcal{D}(\mathfrak{g})^G = U(\mathfrak{s})$ where $\mathfrak{s} = \mathbb{C}h \oplus \mathbb{C}e \oplus \mathbb{C}f$ is the three dimensional simple Lie algebra introduced in the proof of Lemma 2.3. The spaces M^i are the weight spaces of M as a $\mathfrak{s}$-module with respect to the semisimple element $h = -E - \frac{3}{2}$. The eigenvalue of h on M^i is $-i - \frac{1}{2}$. We now see that

$$M = \bigoplus_{k \geq 0} M(-2\rho - k\alpha) \otimes \left(M(-k - \frac{3}{2}) \oplus M(k - \frac{1}{2})\right)$$

as a $(\mathfrak{g}, \mathcal{D}(\mathfrak{g})^G)$-bimodule. Here $M(-k - \frac{3}{2})$ (resp. $M(k - \frac{1}{2})$) is the Verma module for $\mathfrak{s}$ with highest weight $-k - \frac{3}{2}$ (resp. $k - \frac{1}{2}$). For the module $\Gamma^1(M)$ we obtain

$$\Gamma^1(M) = \bigoplus_{k \geq 0} F_{2k} \otimes \left(M(-k - \frac{3}{2}) \oplus M(k - \frac{1}{2})\right)$$

where F_{2k} denotes the irreducible $\mathfrak{g}$-module of dimension $2k+1$. It is interesting to compare this decomposition with the decomposition of the $\mathcal{D}(\mathfrak{g})$-module $\mathcal{O}(\mathfrak{g})$ (with the usual action). The graded character for $\mathcal{O}(\mathfrak{g}) = \bigoplus_i \mathcal{O}^i(\mathfrak{g})$ is given as

$$\begin{aligned}\mathrm{ch}(\mathcal{O}(\mathfrak{g}), q) &= \frac{1}{(1-q)(1-qx)(1-qx^{-1})} \\ &= \sum_{k\geq 0}(x^{-k} + x^{-k+1} + \cdots + x^k) \cdot \frac{q^k}{1-q^2}.\end{aligned}$$

Hence

$$\mathcal{O}(\mathfrak{g}) = \bigoplus_{k\geq 0} F_{2k} \otimes M(-k - \frac{3}{2})$$

as a $(\mathfrak{g}, \mathcal{D}(\mathfrak{g})^G)$-bimodule. The $\mathcal{D}(\mathfrak{g})$-module $\Gamma^1(M)$ seems twice as big as the module $\mathcal{O}(\mathfrak{g})$. In Section 8 we will prove the following

THEOREM 6.1. *Let $Z' = \mathcal{D}(\mathfrak{g})(1 \otimes M(-\frac{3}{2}))$ and $Z'' = \mathcal{D}(\mathfrak{g})(1 \otimes M(-\frac{1}{2}))$. Then Z' and Z'' are irreducible $\mathcal{D}(\mathfrak{g})$-modules and we have a decomposition $\Gamma^1(M) = Z' \oplus Z''$. As $(G, \mathcal{D}(\mathfrak{g})^G)$-bimodules*

$$Z' = \bigoplus_{k\geq 0} F_{2k} \otimes M(-k - \frac{3}{2})$$

and

$$Z'' = \bigoplus_{k\geq 0} F_{2k} \otimes M(k - \frac{1}{2}).$$

Furthermore, Z' is isomorphic with $\mathcal{O}(\mathfrak{g})$ as a $\mathcal{D}(\mathfrak{g})$-module.

7. Duality and a decomposition result

We now introduce another module, $\tilde{M}$, which will play an important role in the study of M. Let $\tilde{M}$ be defined dually to M as

$$\tilde{M} = \mathcal{D}(\mathfrak{g}) \otimes_{S(\mathfrak{h}^* \oplus \mathfrak{n}_- \oplus \mathfrak{n}_+^*)} \mathbb{C}.$$

Let $X_1, \ldots, X_n$ be an orthonormal basis of $\mathfrak{g}$ (with respect to a G-invariant non-degenrate symmetric bilinear form on $\mathfrak{g}$). Let $x_1, \ldots, x_n$ be the corresponding linear coordinates. Define a linear anti-automorphism $D \mapsto D^T$ of $\mathcal{D}(\mathfrak{g})$ by $x_i \mapsto X_i$, $X_i \mapsto x_i$. We assert that there exists a non-degenerate pairing $\langle\ ,\ \rangle$ of M with $\tilde{M}$ which is $\mathcal{D}(\mathfrak{g})$-contravariant, i.e. $\langle Du, v\rangle = \langle u, D^T v\rangle$ for $u \in M, v \in \tilde{M}$. Let $V = \mathfrak{h} \oplus \mathfrak{n}_-^* \oplus \mathfrak{n}_+$. Then $V^* = \mathfrak{h}^* \oplus \mathfrak{n}_- \oplus \mathfrak{n}_+^*$ and $\mathfrak{g} \oplus \mathfrak{g}^* = V \oplus V^*$ and the natural pairing of $S(V)$ with $S(V^*)$ induces a pairing of M with $\tilde{M}$ with the required properties.

LEMMA 7.1. *The non-degenerate pairing of M with $\tilde{M}$ induces a non-degenerate pairing of $\Gamma^j(M)$ with $\Gamma^{2r-j}(\tilde{M})$.*

PROOF. On $\mathcal{O}(G)$ we define an inner product $\langle\ ,\ \rangle$ as follows. Let

$$\mathcal{O}(G) = \bigoplus_{\gamma \in \hat{G}} \mathcal{O}(G)[\gamma]$$

be the decomposition of the G-module $\mathcal{O}(G)$ into isotypic components relative to the action ρ_l (or ρ_r). Let $R : \mathcal{O}(G) \longrightarrow \mathcal{O}(G)[0]$ be the projection onto the invariants $\mathcal{O}(G)^G = \mathbb{C}$. If $f, h \in \mathcal{O}(G)$ then we set $\langle f, h\rangle = R(fh)$. This defines a G-invariant inner product on $\mathcal{O}(G)$. We obtain a product pairing (which we also denote by $\langle\ ,\ \rangle$) of $M \otimes \mathcal{O}(G)$ with $\tilde{M} \otimes \mathcal{O}(G)$. Next we define a non-degenerate pairing of $\wedge^j(\mathfrak{g}/\mathfrak{h})$ with $\wedge^{2r-j}(\mathfrak{g}/\mathfrak{h})$. Let ω_0 be a fixed non-zero element of $\wedge^{2r}(\mathfrak{g}/\mathfrak{h})$. If $\alpha \in \wedge^j(\mathfrak{g}/\mathfrak{h}), \beta \in \wedge^{2r-j}(\mathfrak{g}/\mathfrak{h})$ then we define $\langle \alpha, \beta\rangle$ by $\alpha \wedge \beta = \langle \alpha, \beta\rangle \omega_0$. Let $\xi_1, \ldots, \xi_d$ be a basis of $\wedge^j(\mathfrak{g}/\mathfrak{h})$ and let $\xi^1, \ldots, \xi^d$ be the corresponding dual basis of $\wedge^{2r-j}(\mathfrak{g}/\mathfrak{h})$. Now, if $\omega \in \Gamma^j(M), \eta \in \Gamma^{2r-j}(\tilde{M})$ then we set

$$\langle \omega, \eta\rangle = \sum_i \langle \omega(\xi_i), \eta(\xi^i)\rangle.$$

One checks that this defines a non-degenerate $\mathcal{D}(\mathfrak{g})$-contravariant pairing of $\Gamma^j(M)$ with $\Gamma^{2r-j}(\tilde{M})$. □

We now consider the module $N = \Gamma^r(M)$ and its dual $\tilde{N} = \Gamma^r(\tilde{M})$. The invariants N^G as a $\mathcal{D}(\mathfrak{g})^G$-module are isomorphic with $\mathcal{O}(\mathfrak{h})$ with $\mathcal{D}(\mathfrak{g})^G$ acting on $\mathcal{O}(\mathfrak{h})$ via δ. (This follows from the observation that the trivial infinitesimal character occurs in M only in the direct summand $M(-2\rho) \otimes \mathcal{O}(\mathfrak{h})$.) Similarly, the invariants $\tilde{N}^G$ are isomorphic with $S(\mathfrak{h})$. The pairing of N with $\tilde{N}$ induces a pairing of N^G with $\tilde{N}^G$.

In [Wa1] we studied the module $\mathcal{O}(\mathfrak{h})$ as a $(\mathcal{D}(\mathfrak{h})^W, W)$-bimodule. We record the main result. Let $\hat{W}$ denote the set of equivalence classes of irreducible finite-dimensional representations of W. If $\gamma \in \hat{W}$ we fix $V_\gamma \in \gamma$. To each $\gamma \in \hat{W}$ there corresponds a $\mathcal{D}(\mathfrak{h})^W$-module V^γ such that

i) If V^γ is isomorphic with $V^{\gamma'}$ as a $\mathcal{D}(\mathfrak{h})^W$-module then $\gamma = \gamma'$.

ii) As a $(\mathcal{D}(\mathfrak{g})^W, W)$-bimodule $\mathcal{O}(\mathfrak{h})$ is isomorphic with

$$\bigoplus_{\gamma \in \hat{W}} V^\gamma \otimes V_\gamma.$$

LEMMA 7.2. *Let V be an irreducible $\mathcal{D}(\mathfrak{g})^G$-module. Then, up to equivalence, there exists exactly one irreducible $(\mathcal{D}(\mathfrak{g}), G)$-module $L(V)$ such that $L(V)^G$ is isomorphic with V as a $\mathcal{D}(\mathfrak{g})^G$-module.*

PROOF. We define a $(\mathcal{D}(\mathfrak{g}), G)$-module, $M(V)$, as follows:

$$M(V) = \mathcal{D}(\mathfrak{g}) \otimes_{\mathcal{D}(\mathfrak{g})^G} V$$

with G acting on the first factor. Then $M(V)^G = 1 \otimes V$. Clearly, if X is any irreducible $(\mathcal{D}(\mathfrak{g}), G)$-module such that X^G is isomorphic with V as a $\mathcal{D}(\mathfrak{g})^G$-module then X is isomorphic with a quotient of $M(V)$. We assert that $M(V)$

contains a unique proper maximal $(\mathcal{D}(\mathfrak{g}), G)$-submodule. Set

$$\mathcal{Z} = \{Z \subset M(V) \mid Z \text{ is a } (\mathcal{D}(\mathfrak{g}), G)\text{-submodule such that } Z^G = 0\}.$$

If $Z_1, Z_2 \in \mathcal{Z}$ then $Z_1 + Z_2 \in \mathcal{Z}$. Hence

$$M'(V) = \sum_{Z \in \mathcal{Z}} Z$$

is a proper $(\mathcal{D}(\mathfrak{g}), G)$-submodule of $M(V)$. We note that if X is a $(\mathcal{D}(\mathfrak{g}), G)$-submodule of $M(V)$ such that $X^G \neq 0$ then $X^G \supset 1 \otimes V$, so $X = M(V)$. Hence $M'(V)$ is the unique proper maximal $(\mathcal{D}(\mathfrak{g}), G)$-submodule of $M(V)$. Now, up to equivalence,

$$L(V) = M(V)/M'(V)$$

is the only irreducible $(\mathcal{D}(\mathfrak{g}), G)$-module such that $L(V)^G$ is isomorphic with V as a $\mathcal{D}(\mathfrak{g})^G$-module. □

REMARK. If X is a compatible $(\mathcal{D}(\mathfrak{g}), G)$-module with G connected then X is irreducible as a $(\mathcal{D}(\mathfrak{g}), G)$-module iff it is as a $\mathcal{D}(\mathfrak{g})$-module.

We define a compatible $(\mathcal{D}(\mathfrak{g}), G)$-module as follows. Let $N' = \mathcal{D}(\mathfrak{g})N^G/R$ with $R = \{n \in \mathcal{D}(\mathfrak{g})N^G \mid \langle n, \mathcal{D}(\mathfrak{g})\tilde{N}^G\rangle = 0\}$. Note that $N' = N$ if $\mathcal{D}(\mathfrak{g})N^G = N$ and $\mathcal{D}(\mathfrak{g})\tilde{N}^G = \tilde{N}$.

THEOREM 7.3. *The compatible $(\mathcal{D}(\mathfrak{g}), G)$-module N' is isomorphic with*

$$\bigoplus_{\gamma \in \hat{W}} (\dim \gamma) N^\gamma$$

where the N^γ are irreducible pairwise non-isomorphic $\mathcal{D}(\mathfrak{g})$-modules. Furthermore, $(N^\gamma)^G$ is isomorphic with V^γ as a $\mathcal{D}(\mathfrak{g})^G$-module.

PROOF. Let $N^G = V_1 \oplus \cdots \oplus V_w$ be a decompostion of the $\mathcal{D}(\mathfrak{g})^G$-module N^G into irreducible submodules. We set $N_i = \mathcal{D}(\mathfrak{g})V_i$. Clearly, $N_i^G = \mathcal{D}(\mathfrak{g})^G V_i = V_i$. Let N_i' denote the image of N_i in N'. Since $R \cap N^G = 0$ we still have that $(N_i')^G$ is isomorphic with V_i as a $\mathcal{D}(\mathfrak{g})^G$-module. Let $X \subset N_i'$ be a proper $\mathcal{D}(\mathfrak{g})$-submodule. Then X^G is isomorphic with a proper $\mathcal{D}(\mathfrak{g})^G$-submodule of V_i. Hence $X^G = 0$. This implies $X \perp \tilde{N}^G$, so $X \perp \mathcal{D}(\mathfrak{g})\tilde{N}^G$. Hence $X = 0$. Thus, the modules N_i' are irreducible and the sum $\sum_i N_i' = N'$ is direct. The result now follows from the observations above. □

8. Conjectures

We will first study the consequences of the following

CONJECTURE 8.1.

i) *The module M is $\mathfrak{n}_-$-free.*
ii) *The module $\tilde{M}$ is $\mathfrak{n}_+$-free.*

REMARK. We proved this conjecture for the simple Lie algebras of type $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ and $\mathbf{B}_2 = \mathbf{C}_2$. (The proof for $\mathbf{B}_2 = \mathbf{C}_2$ is a computer proof.) However, our technique breaks down for $\mathbf{G}_2$.

THEOREM 8.2. *Assume that Conjecture 8.1 is true. Then* $\Gamma^j(M) = 0$ *for* $j \neq r$.

PROOF. One can show that if M is free as an $\mathfrak{n}_-$-module then M admits a Verma composition series, i.e. there exists a sequence of $\mathfrak{g}$-submodules $0 = M_{-1} \subset M_0 \subset M_1 \subset \cdots$ such that $M = \bigcup_i M_i$ and each M_i/M_{i-1} is a Verma module. This implies $\Gamma^j(M) = 0$ for $j < r$ (cf. [RRG, Lemma 6.4.2]). Similarly, if $\tilde{M}$ is free as an $\mathfrak{n}_+$-module then $\tilde{M}$ admits a Verma composition series (relative to the opposite Borel subalgebra). Hence $\Gamma^j(\tilde{M}) = 0$ for $j < r$. So $\Gamma^j(M) = 0$ for $j > r$ by Lemma 7.1. □

We now study the non-zero module $N = \Gamma^r(M)$ as a $\mathcal{D}(\mathfrak{g})$-module. First we recall some notions. Let V be a vector space over $\mathbb{C}$ of dimension $n < \infty$. The Bernstein filtration of $\mathcal{D}(V)$ is defined by

$$\mathcal{D}(V)_k = \sum_{i+j \leq k} S^i(V^*)S^j(V).$$

With this filtration $\mathcal{D}(V)$ is a filtered algebra with corresponding graded algebra, $\operatorname{gr}\mathcal{D}(V)$, isomorphic with $\mathcal{O}(V \times V^*)$. Let M be a $\mathcal{D}(V)$-module. A filtration $M_0 \subset M_1 \subset \cdots$ of M is called a $\mathcal{D}(V)$-filtration if $\mathcal{D}(V)_i M_j \subset M_{i+j}$. To each filtered $\mathcal{D}(V)$-module M there is an associated graded module $\operatorname{gr} M$ over $\operatorname{gr}\mathcal{D}(V)$. If M admits a $\mathcal{D}(V)$-filtration such that $\operatorname{gr} M$ is a finitely generated $\operatorname{gr}\mathcal{D}(V)$-module (a good filtration) and such that

$$\dim M_k = \frac{C}{n!}k^n + \text{ terms of lower order}$$

then M is called holonomic. The following result is due to Bernstein (see [Eh, Prop. 1.12 and 1.15] for a proof).

THEOREM 8.3. *Suppose that M admits a (not necessarily good) $\mathcal{D}(V)$-filtration such that*

$$\dim M_k \leq \frac{C}{p!}k^p \qquad \text{for } k \gg 0.$$

If $p < n$ then $M = 0$ and if $p = n$ then M is holonomic and the length of the module M is bounded by C.

We return to the case $V = \mathfrak{g}$ of the previous sections. Let M and $\tilde{M}$ be the modules defined in Sections 4 and 7. Our goal is to prove that the module $N = \Gamma^r(M)$ is holonomic (under the assumption that Conjecture 8.1 holds). We fix some notation. Let $\alpha_1, \ldots, \alpha_r$ be the postive roots and assume that $\alpha_1, \ldots, \alpha_l$ are the simple roots. Set $Q = \sum_{i=1}^l \mathbb{Z}\alpha_i$ and $Q_+ = \sum_{i=1}^l \mathbb{N}\alpha_i$. For $\lambda = \sum_{i=1}^l n_i\alpha_i \in Q$ then the number $\operatorname{ht}\lambda = \sum_{i=1}^l n_i$ is called the height of λ. If $K \in \mathbb{N}^r$ then we define $\langle K \rangle = \sum_{i=1}^r k_i\alpha_i$. We note that $|K| = \sum_{i=1}^r k_i \leq \operatorname{ht}\langle K \rangle$.

LEMMA 8.4. $\dim M^i_{-2\rho-\lambda} \leq C(|i| + \operatorname{ht}\lambda)^{2r-1}$.

PROOF. From Lemma 4.1 we obtain

$$\begin{aligned}\dim M^d_{-2\rho-\lambda} &= \#\{(I,J,K) \in \mathbb{N}^l \times \mathbb{N}^r \times \mathbb{N}^r \mid \langle J\rangle + \langle K\rangle = \lambda,\ i = |I| + |J| - |K|\} \\ &= \sum_{p=0}^{|d|+\operatorname{ht}\lambda} \#\{I \in \mathbb{N}^l \mid |I| = p\} \\ &\qquad \times \#\{(J,K) \in \mathbb{N}^r \times \mathbb{N}^r \mid \langle J\rangle + \langle K\rangle = \lambda,\ i = p + |J| - |K|\}\end{aligned}$$

Clearly, $\#\{I \in \mathbb{N}^l \mid |I| = p\} \leq (p+1)^{l-1}$. Similarly, for each p we obtain the estimate

$$\begin{aligned}\#\{(J,K) \in &\mathbb{N}^r \times \mathbb{N}^r \mid \langle J\rangle + \langle K\rangle = \lambda,\ i = p + |J| - |K|\} \\ &= \sum_{\langle L\rangle = \lambda} \#\{(J,K) \in \mathbb{N}^r \times \mathbb{N}^r \mid J + K = L,\ i = p + |J| - |K|\} \\ &\leq \sum_{\langle L\rangle = \lambda} (\operatorname{ht}\lambda + 1)^{r-1} \\ &\leq (\operatorname{ht}\lambda + 1)^{r-l}(\operatorname{ht}\lambda + 1)^{r-1} = (\operatorname{ht}\lambda + 1)^{2r-l-1}\end{aligned}$$

Thus,

$$\dim M^i_{-2\rho-\lambda} \leq ((|i| + \operatorname{ht}\lambda) + 1)^l (\operatorname{ht}\lambda + 1)^{2r-l-1} \leq C(|i| + \operatorname{ht}\lambda)^{2r-1}.$$

This completes the proof. □

THEOREM 8.5. *Assume that Conjecture 8.1 is true. Then N is a holonomic $\mathcal{D}(\mathfrak{g})$-module.*

PROOF. If $\Lambda \in Q$ is dominant then we fix a finite dimensional irreducible $(\mathfrak{g}, G)$-module F_Λ with highest weight Λ. This defines a bijection of the set of dominant weights in Q with the set of equivalence classes of irreducible finite-dimensional $(\mathfrak{g}, G)$-modules. Let $N^i[\Lambda]$ denote the Λ-isotypic component of the $(\mathfrak{g}, G)$-module $N^i = \Gamma^r(M^i)$. We note that

$$(\mathfrak{g} \oplus \mathfrak{g}^*)N^i[\Lambda] \subset \sum_{\alpha \in \Phi \cup \{0\}} (N^{i+1}[\Lambda + \alpha] + N^{i-1}[\Lambda + \alpha]).$$

Let β be the highest root. Then we define

$$N_k = \sum_{|i|, \operatorname{ht}\Lambda \leq k \operatorname{ht}\beta} N^i[\Lambda].$$

The sequence $N_0 \subset N_1 \subset \cdots$ is a $\mathcal{D}(\mathfrak{g})$-filtration of N. We investigate the dimensions of the spaces N_k. First we observe that

$$\dim N_k = \sum_{|i|, \operatorname{ht}\Lambda \leq k \operatorname{ht}\beta} \dim F_\Lambda \cdot \dim \operatorname{Hom}_G(F_\Lambda, N^i).$$

From Weyl's formulas we derive the estimate $\dim(F_\Lambda) \leq C_1(\operatorname{ht}\Lambda)^r$. Thus,

$$\dim N_k \leq C_1 k^r \cdot \sum_{|i|,\operatorname{ht}\Lambda \leq k \operatorname{ht}\beta} \dim \operatorname{Hom}_G(F_\Lambda, N^i).$$

We now consider the characters of the modules M^i. Under our assumption we may write

$$\operatorname{ch} M^i = \sum_{\lambda \in Q_+} a_{i,\lambda} \operatorname{ch} M(-2\rho - \lambda)$$

with $a_{i,\lambda} \in \mathbb{N}$. If $\Lambda \in Q$ is dominant then $a_{i,\Lambda}$ gives an upper bound for $\dim \operatorname{Hom}_G(F_\Lambda, N^i)$. Set $a_{ij} = \sum_{\operatorname{ht}\lambda = j} a_{i,\lambda}$. Then

$$\dim N_k \leq C_1 k^r (2k \operatorname{ht}\beta + 1) \max_{|i| \leq k \operatorname{ht}\beta} \Big(\sum_{j=0}^{k \operatorname{ht}\beta} a_{ij} \Big).$$

Before we can complete the proof we need some preparation. We filter the modules M^i as follows. Set $M^i_p = \sum_{\operatorname{ht}\mu \leq p \operatorname{ht}\beta} M^i_{-2\rho-\mu}$. Similarly, for each Verma module $M(\lambda)$ set $M(\lambda)_p = \sum_{\operatorname{ht}\mu \leq p \operatorname{ht}\beta} M(\lambda)_{\lambda-\mu}$.

CLAIM. $\dim M(\lambda)_p \geq C_2 p^r$.

PROOF OF THE CLAIM. Set $M(\lambda)_{(0)} = M(\lambda)_\lambda$ and for $p > 0$ inductively $M(\lambda)_{(p)} = M(\lambda)_{(p-1)} + \mathfrak{g} M(\lambda)_{(p-1)}$. Clearly, $M(\lambda)_{(p)} \subset M(\lambda)_p$. One checks that

$$\dim M(\lambda)_{(p)} = \frac{1}{r!} p^r + \text{ terms of lower order} \geq C_2 p^r.$$

The claim follows.

Now,

$$\begin{aligned} \dim M^i_p &= \sum_{\lambda \in Q_+} a_{i,\lambda} \dim M(-2\rho - \lambda)_{p - \operatorname{ht}\lambda} \\ &\geq C_2 \sum_{\lambda \in Q_+} a_{i,\lambda} (p - \operatorname{ht}\lambda)^r = C_2 \sum_{j=0}^{p} a_{ij} (p-j)^r. \end{aligned}$$

Replacing p by $2p$ we find

$$\dim M^i_{2p} \geq C_2 \sum_{j=0}^{2p} a_{ij} (2p-j)^r \geq C_2 \sum_{j=0}^{p} a_{ij} (2p-j)^r \geq C_2 \sum_{j=0}^{p} a_{ij} p^r.$$

On the other hand, in light of Lemma 8.4, we have the estimate

$$\max_{|i| \leq p} (\dim M^i_{2p}) \leq C_3 p^{2r+l-1}.$$

Thus,

$$\max_{|i| \leq p} \Big(\sum_{j=0}^{p} a_{ij} \Big) \leq C_4 p^{r+l-1}.$$

We finally obtain

$$\dim N_k \leq C_1 k^r (2k \operatorname{ht} \beta + 1) C_4 (k \operatorname{ht} \beta)^{r+l-1} \leq C k^{2r+l}.$$

Since $2r + l = \dim \mathfrak{g}$ we conclude that N is holonomic by Theorem 8.3. □

Next we introduce another conjecture.

CONJECTURE 8.6.

i) $\mathcal{D}(\mathfrak{g})N^G = N$.
ii) $\mathcal{D}(\mathfrak{g})\tilde{N}^G = \tilde{N}$.

Theorem 7.3 implies

THEOREM 8.7. *Assume that Conjecture 8.6 is true. Then the compatible* $(\mathcal{D}(\mathfrak{g}), G)$*-module* N *is isomorphic with*

$$\bigoplus_{\gamma \in \hat{W}} (\dim \gamma) N^\gamma$$

where the N^γ *are irreducible pairwise non-isomorphic* $\mathcal{D}(\mathfrak{g})$*-modules. Furthermore,* $(N^\gamma)^G = V^\gamma$ *as a* $\mathcal{D}(\mathfrak{g})^G$*-module.*

REMARK. The assertion in Theorem 8.7 is actually equivalent to Conjecture 8.6.

We now show that Conjecture 8.6 is true for $\mathfrak{g} = \mathfrak{sl}_2$.

PROOF OF THEOREM 6.1. We have seen in Section 6 that

$$N = \bigoplus_{k \geq 0} F_{2k} \otimes (M(-k - \frac{3}{2}) \oplus M(k - \frac{1}{2})).$$

Let $Z^{\mathrm{sgn}} = \mathcal{D}(\mathfrak{g})(1 \otimes M(-\frac{3}{2}))$ and $Z^1 = \mathcal{D}(\mathfrak{g})(1 \otimes M(-\frac{1}{2}))$. The irreducible $\mathcal{D}(\mathfrak{g})$-module N^{sgn} (see Section 7) is a quotient of Z^{sgn}. We note that $N^{\mathrm{sgn}} = \mathcal{O}(\mathfrak{g})$ as a $\mathcal{D}(\mathfrak{g})$-module. In Section 6 we have also seen that

$$N^{\mathrm{sgn}} = \bigoplus_{k \geq 0} F_{2k} \otimes M(-k - \frac{3}{2})$$

as a $(G, \mathcal{D}(\mathfrak{g})^G)$-bimodule. This implies

$$Z^{\mathrm{sgn}} \supset \bigoplus_{k \geq 0} F_{2k} \otimes M(-k - \frac{3}{2})$$

and (since N^{sgn} is irreducible)

$$Z^1 \subset \bigoplus_{k \geq 0} F_{2k} \otimes M(k - \frac{1}{2}).$$

Thus, we may write

$$Z^1 = \bigoplus_{k \in S} F_{2k} \otimes M(k - \frac{1}{2})$$

with $S \subset \{0, 1, 2, \dots\}$. We now observe that

$$(\mathfrak{g} \oplus \mathfrak{g}^*) F_{2k} \otimes M(k - \frac{1}{2})$$

involves the G-types F_{2k+2}, F_{2k} and F_{2k-2}. Thus if $k_0 \in S$ and $k_0 + 1 \notin S$ then the subspace

$$Z = \bigoplus_{k \in S, k \leq k_0} F_{2k} \otimes M(k - \frac{1}{2})$$

is $\mathcal{D}(\mathfrak{g})$-invariant. If we use the same filtration of N as in the proof of Theorem 8.5 we find that $\dim Z_j \leq Cj^2$. (Here $Z_j = N_j \cap Z$.) Theorem 8.3 would imply that $Z = 0$. Hence if $k_0 \in S$ then $k_0 + 1 \in S$. Since $0 \in S$ we see that $S = \{0, 1, 2, \dots\}$. This implies $N = Z^{\text{sgn}} + Z^1 = \mathcal{D}(\mathfrak{g}) N^G$. The same arguments replacing $\mathcal{O}(\mathfrak{g})$ with $S(\mathfrak{g})$ prove the assertion for $\tilde{N}$. □

REMARK. Sam Evens has shown that the results in [HK] imply that the assertions in Theorem 8.2, Theorem 8.5 and Theorem 8.7 are true for a module V whose construction is geometric and analogous to our construction of N. Thus, if we can show that V is isomorphic with N then the consequences of our conjectures would be true.

REFERENCES

[Br] G. Brown, *A remark on semisimple Lie algebras*, Proc. Amer. Math. Soc. **15** (1964), 518.

[BW] A. Borel, N. R. Wallach, *Continuous Cohomology, Discrete Subgroups and Representations of Reductive Groups*, Annals of Math. Studies, Study 94, Princeton University Press, Princeton, NJ, 1980.

[Eh] F. Ehlers, *The Weyl algebra*, In: *Algebraic D-modules*, Chapter V, *Perspect. Math.*, Vol. 2, Academic Press, Boston, MA, 1987.

[HC1] Harish-Chandra, *Differential operators on a semisimple Lie algebra*, Amer. J. Math. **79** (1957), 87-120.

[HC2] ———, *Invariant differential operators and distributions on a semisimple Lie algebra*, Amer. J. Math. **86** (1964), 534-564.

[HK] R. Hotta, M. Kashiwara, *The invariant holonomic system on a semisimple Lie algebra*, Invent. Math. **75** (1984), 327-358.

[LS] T. Levasseur, J. T. Stafford, *Invariant differential operators and an homomorphism of Harish-Chandra*, to appear, J. Amer. Math. Soc..

[RRG] N. R. Wallach, *Real Reductive Groups, I*, Academic Press, Boston, MA,, 1988.

[Wa1] ———, *Invariant differential operators on a reductive Lie algebra and Weyl group representations*, J. Amer. Math. Soc. **6** (1993), 779-816.

[Wa2] ———, *Transfer of unitary representations between real forms*, to appear, Contemp. Math..

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Spectral Zeta Series of Rank 1 Space Forms

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Dedicated to Ray A. Kunze

ABSTRACT. We give an alternative proof of Weyl's law (Gelfand's Conjecture) on the asymptotics of the class 1 spectrum, and we give a factorization of Selberg's zeta function.

1. Introduction

Let $X_\Gamma = \Gamma \backslash X$ be a compact space form of an irreducible rank 1 symmetric space $X = G/K$ of noncompact type where K is a maximal compact subgroup of the simple group G and Γ is a discrete subgroup of G. Fix a finite-dimensional unitary representation χ of Γ and let χ also denote the corresponding character. Then the induced unitary representation $\pi_\chi = \mathrm{ind}_{\Gamma\uparrow G}\chi$ of G decomposes discretely:

$$\pi_\chi = \sum_{\pi \in \hat{G}} m_\pi(\Gamma)\pi, \tag{1.1}$$

$\hat{G}$ denoting the unitary dual of G, where each multiplicity $m_\pi(\Gamma) \geq 0$ is finite. Let $\{\pi_j\}_{j\geq 0} \subset \hat{G}$ be the subset of all class 1 representations (with respect to K) such that $n_j(\chi) \stackrel{\text{def}}{=} m_{\pi_j}(\Gamma) > 0$. Attached to the class 1 spectrum $\{\pi_j, n_j(\chi)\}_{j\geq 0}$ of π_χ is a spectral zeta series of Minakshisundaram-Pleijel type

$$D_\Gamma(s; b, \chi) = \sum_{j=0}^{\infty} \frac{n_j(\chi)}{(b + \lambda_j)^s} \tag{1.2}$$

with parameter $b > 0$ where $\{\lambda_j\}_{j\geq 0}$ is the spectrum of the Laplacian $-\Delta_\Gamma$ on smooth sections of the vector bundle over X_Γ induced by χ. The eigenvalues λ_j

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depend on χ and are described in (2.6) below. $D_\Gamma(s; b, \chi)$ converges absolutely for $\mathbf{Re}s > d/2$, $d = \dim X$; cf. [10, 14, 19]. We assume X_Γ is a manifold.

We find an explicit meromorphic continuation to the whole complex plane of $s \to D_\Gamma(s; b, \chi)$ in terms of the harmonic analysis of X. The result, Theorem 3.7, is simpler and more general than that obtained in [21]. Combining Theorem 3.7 with a Tauberian theorem of Delange we obtain a quick proof of a Weyl asymptotic law for the class 1 multiplicities $\{n_j(\chi)\}_{j\geq 0}$ (see Theorem 3.10) as conjectured, in principle, by Gelfand [11] and first proved by Gangolli [7, 8]. Eaton [5] and Wallach [19] also have obtained proofs where, more generally, Γ is allowed to have torsion (as in [7]). On the other hand our method of proof extends, in principle, to noncompact finite volume quotients X_Γ quite naturally (and to Γ with torsion; see remarks following Theorem 3.15), given the theory of Eisenstein series. We do hope to push further in this direction. Theorem 3.7 has other applications. It leads, for example, to a general factorization of Selberg's zeta function attached to X_Γ; cf. Theorem 3.15.

In 1972, the author wrote of his thesis advisor Ray Kunze: "The debts we owe him for encouraging and guiding our career, over a prolonged period of time, will extend through future years. So will our deep admiration of him." Some 20 plus years later, these words indeed continue to echo truth. With much affection we dedicate this paper to Ray.

2. Some Further Notation

Let $G = KA_pN$ be an Iwasawa decomposition of G and let g_0, a_p, n_0 denote the Lie algebras of G, A_p, N. For θ the Cartan involution of G (with respect to K), $g = g_0^{\mathbf{C}}$ the complexification of g_0, and $a \supset a_p$ a θ-stable Cartan subalgebra of g_0, let Φ^+ be an a_p-compatible system of positive roots in the set Φ of nonzero roots of $(g, a^{\mathbf{C}})$. Then for the set of restrictions $\Sigma^+ = \{\alpha|a_p|\alpha \in P^+\}$ where $P^+ = \{\alpha \in \Phi^+|\alpha \not\equiv 0 \text{ on } a_p\}$ one has (as $\dim a_p = 1$)

$$n_0 = \sum_{\alpha\in\Sigma^+} g_\alpha, \quad \Sigma^+ = \{\beta\} \text{ or } \{\beta, 2\beta\} \tag{2.1}$$

with g_α a (real) restricted root space. We normalize a choice of basis vector H_0 of a_p by $\beta(H_0) = 1$ and we set

$$\rho = \frac{1}{2}\sum_{\alpha\in P^+} \alpha, \quad \rho_0 = \rho(H_0). \tag{2.2}$$

Let $H : G \to a_p$ be the smooth map such that each $x \in G$ has an Iwasawa decomposition $x = k(x) \exp H(x) n(x) \in KA_pN$.

With the above notation in place we now specify the normalization of Haar

measures used henceforth: For compactly supported continuous functions f, h, ψ

$$\int_{A_p} f(a)da = \int_{\mathbf{R}} f(\exp tH_0)dt, \qquad \int_N e^{-2\rho(H(\theta n))}dn = 1,$$

$$\int_G h(x)dx = \int_N \int_{A_p} \int_K h(kan)e^{2\rho(\log a)}dkdadn, \tag{2.3}$$

$$\int_{a_p^*} \psi(\nu)d\nu = \frac{1}{2\pi}\int_{\mathbf{R}} \psi(t\beta)dt,$$

where a_p^* is the dual space of a_p, dt is Lebesgue measure on $\mathbf{R}$, and dk is normalized Haar measure on K. As G, Γ are unimodular there is, given dx, a unique G-invariant measure m_Γ on $\Gamma \setminus G$ such that for $h \in C_c(G)$

$$\int_G h(x)dx = \int_{\Gamma\setminus G} \left[\sum_{\gamma\in\Gamma} h(\gamma x)\right] dm_\Gamma(\Gamma x). \tag{2.4}$$

Write $\mathrm{vol}(\Gamma \setminus G) = \int_{\Gamma\setminus G} 1 dm_\Gamma$ as usual.

Each of the class 1 representations π_j of section 1 corresponds to a positive definite spherical function ϕ_j which has the form $\phi_j = \phi_{\nu_j}$ for some $\nu_j \in a_p^{*,\mathbf{C}}$ where for $\nu \in a_p^{*,\mathbf{C}}, x \in G$, by [1],

$$\phi_\nu(x) = \int_K e^{(i\nu-\rho)(H(xk))}dk. \tag{2.5}$$

Since π_j depends on χ, ν_j does also and we may write $\nu_j = \nu_j(\chi)$. We take $\pi_0 =$ the trivial representation: $\nu_0 = i\rho$. The π_j with $\nu_j(H_0) \in i\mathbf{R} - \{0\}$ are finite in number and belong to the spherical complementary series. The eigenvalues λ_j in (1.2) of $-\Delta_\Gamma$ are given by

$$\lambda_j = \lambda_j(\chi) = \rho_0^2 + \nu_j(\chi)^2(H_0). \tag{2.6}$$

Therefore $\lambda_0 = 0$ and we take $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots$; $\lim_{j\to\infty} \lambda_j = 0$.

Finally we introduce some notation used in the trace formula. As Γ is cocompact and torsion free one can find for each $\gamma \in \Gamma - \{1\}$ $t_\gamma > 0$ and $m_\gamma \in M$, the centralizer of A_p in K, such that γ is G-conjugate to $m_\gamma \exp t_\gamma H_0$, with m_γ unique up to conjugation in M. By [20] in fact

$$\begin{aligned} e^{t_\gamma} &= \max\{|c| : c = \text{ an eigenvalue of } \mathrm{Ad}(\gamma) : g \to g\} \\ &\text{if } \Sigma^+ = \{\beta\} \text{ in (2.1), with } |c| \text{ replaced by } |c|^{\frac{1}{2}} \text{ if } \Sigma^+ = \{\beta, 2\beta\}; \\ &\text{see Table 1 below.} \end{aligned} \tag{2.7}$$

The following function C on $\Gamma - \{1\}$ is therefore well-defined:

$$C(\gamma)^{-1} = e^{t_\gamma\rho_0}|\det_{n_0}(\mathrm{Ad}(m_\gamma \exp t_\gamma H_0)^{-1} - 1)| \tag{2.8}$$

Also each $\gamma \in \Gamma - \{1\}$ has a unique representation $\gamma = \delta^{j(\gamma)}$ where $\delta \in \Gamma - \{1\}$ is a primitive element and $j(\gamma) \geq 1$ is an integer; i.e. δ cannot be expressed as

γ_1^j with $\gamma_1 \in \Gamma, j > 1$ an integer. Let C_Γ be a complete set of representatives in Γ of its conjugacy classes.

3. Meromorphic Continuation of Zeta

Following the method of Randol [15], who considered the case $G = SL(2, \mathbf{R}) \approx SO_1(2,1)$ with $\chi = 1$, we can find for any split rank 1 simple Lie group G as above an explicit meromorphic continuation to $\mathbf{C}$ of $s \to D_\Gamma(s; b, \chi)$, using Selberg's trace formula and explicit knowledge of the Plancherel measure $|c(r)|^{-2}dr$ of X where $c(\cdot)$ is Harish-Chandra's c-function.

Up to local isomorphism we take $G = SO_1(2n,1)$ or $SO_1(2n+1,1)$ with $n \geq 1$, $SU(n,1)$ or $SP(n,1)$ with $n \geq 2$, or $F_{4(-20)}$. Given $z \in \mathbf{C}$ fixed define the following polynomials $\Pi(r; z)$ in r of degree $(d-2)/2$ for $G \neq SO_1(2n+1,1)$ and of degree $(d-1)/2 = n$ for $G = SO_1(2n+1;1)$:

$$\Pi(r;z) = \prod_{j=2}^{n} \left[r + \left(n - j + \frac{1}{2}\right)^2 - z \right] \text{ for } G = SO_1(2n,1)\ (n \geq 1)$$

$$\Pi(r;z) = \prod_{j=1}^{n-1} \left[\frac{r + (n-2j)^2 - z}{4} \right] \text{ for } G = SU(n,1)\ (n \geq 2)$$

$$\Pi(r;z) = \left[\frac{r+1-z}{4} \right] \prod_{j=3}^{n+1} \left[\frac{r}{4} + \left(n - j + \frac{3}{2}\right)^2 - \frac{z}{4} \right]$$

$$\times \left[\frac{r}{4} + \left(n - j - \frac{5}{2}\right)^2 - \frac{z}{4} \right] \text{ for } G = SP(n,1)\ (n \geq 2) \tag{3.1}$$

$$\Pi(r;z) = \left[\frac{r}{4} + \left(\frac{1}{2}\right)^2 - \frac{z}{4} \right]^2 \left[\frac{r}{4} + \left(\frac{3}{2}\right)^2 - \frac{z}{4} \right]^2 \left[\frac{r}{4} + \left(\frac{5}{2}\right)^2 - \frac{z}{4} \right]$$

$$\times \left[\frac{r}{4} + \left(\frac{7}{2}\right)^2 - \frac{z}{4} \right] \left[\frac{r}{4} + \left(\frac{9}{2}\right)^2 - \frac{z}{4} \right] \text{ for } G = F_{4(-20)}$$

$$\Pi(r;z) = \prod_{j=1}^{n} \left[r + (n-j)^2 - z \right] \text{ for } G = SO_1(2n+1,1)\ (n \geq 1).$$

By definition, $\Pi(r^2 + z; z)$ is an even polynomial $P(r)$ in r, independent of z, of degree $d-2$ for $G \neq SO_1(2n+1,1)$, and of degree $d-1 = 2n$ for $G = SO_1(2n+1,1)$. With our normalization of measures, Miatello's computation

[13] of Harish-Chandra's Plancherel density takes the form

$$(3.2)\quad |c(r)|^{-2} = \begin{cases} C_G \pi r P(r) \tanh \pi r & \text{for } G = SO_1(2n,1) \\ C_G \pi r P(r) \begin{bmatrix} \tanh \frac{\pi}{2} r \\ \text{or} \\ \coth \frac{\pi}{2} r \end{bmatrix} & \text{for } G = SU(n,1), \text{ with the cotangent choice for } n \text{ even} \\ C_G \pi r P(r) \tanh \frac{\pi}{2} r & \text{for } G = SP(n,1) \text{ or } F_{4(-20)} \\ C_G \pi P(r) & \text{for } G = SO_1(2n+1,1) \end{cases}$$

where

$$C_G = (2^{4n-4}\Gamma(n)^2)^{-1}, (2^{2n-1}\Gamma(n)^2)^{-1}, (2^{4n+1}\Gamma(2n)^2)^{-1},$$
$$(3.3)\qquad (2^{21}\Gamma(8)^2)^{-1}, (2^{4n-2}\Gamma(n+\frac{1}{2})^2)^{-1}$$

in the respective cases.

Our interest is in the case $z = b + \rho_0^2$ with $b \in \mathbf{C}$ fixed. We define coefficients $m_j(b)$ by

$$(3.4)\qquad \Pi(r; b+\rho_0^2) = \sum_{j=0}^{m} m_j(b) r^j$$

where $m = (d-2)/2$ for $G \neq SO_1(2n+1,1)$, $m = (d-1)/2 = n$ for $G = SO_1(2n+1,1)$. ρ_0 is given in Table 1.

Table 1

G (up to local isomorphism)	ρ_0	d	Σ^+	$a(G)$	$m_{\frac{d}{2}-1}(b)$
$SU(n,1)$, $n \geq 2$	n	$2n$	$\{\beta, 2\beta\}$	$\frac{\pi}{2}$	$\frac{1}{4^{n-1}}$
$SP(n,1)$, $n \geq 2$	$2n+1$	$4n$	$\{\beta, 2\beta\}$	$\frac{\pi}{2}$	$\frac{1}{4^{2(n-1)}}$
$SO_1(2n,1)$, $n \geq 1$	$n-\frac{1}{2}$	$2n$	$\{\beta\}$	π	1
$SO_1(2n+1,1)$, $n \geq 1$	n	$2n+1$	$\{\beta\}$		$m_n(b) = 1$
$F_{4(-20)}$	11	16	$\{\beta, 2\beta\}$	$\frac{\pi}{2}$	$\frac{1}{4^7}$

One more definition is needed:

$$(3.5)\qquad \theta_\Gamma(t; b) = \frac{1}{\sqrt{4\pi t}} \sum_{\gamma \in C_\Gamma - \{1\}} \chi(\gamma) t_\gamma j(\gamma)^{-1} C(\gamma) e^{-\left(bt + \rho_0^2 t + t_\gamma^2/4t\right)}$$

for $t, b > 0$; see definitions (2.7), (2.8). The behavior of $\theta_\Gamma(t; b)$ at $0, \infty$ is given by

LEMMA 3.6. *$\exists$ constants $\varepsilon, \alpha, C(b) > 0 \ni \lim_{t\to 0^+} \theta_\Gamma(t;b)e^{\varepsilon/t} = 0$, $|\theta_\Gamma(t;b) - n_0(\chi)e^{-bt}| \le C(b)e^{-\alpha t}$ for $t \ge 1$.*

The first statement requires the asymptotic result $\lim_{x\to\infty} \beta x e^{-\beta x}E(x) = 1$ for suitable $\beta > 0$ where $E(x) = |\{\gamma \in C_\Gamma - \{1\} | t_\gamma \le x\}|$; see [2, 8]. One can in fact take $\beta = 2\rho_0$. Lemma 3.6 is proved in [21] in the case $\chi = 1$. The same proof works for general χ.

Here's the main theorem, where we exclude the cotangent case in (3.2) for simplicity of statement. For the latter case see remarks following Theorem 3.15.

THEOREM 3.7. *For $b > 0$ let $T_\Gamma(s;b,\chi) = \int_0^\infty \theta_\Gamma(t;b)t^{s-1}dt$, which is an entire function of s by Lemma 3.6. For $a > 0$ let $K(s;b,a) = \int_{\mathbf{R}}(b + \rho_0^2 + x^2)^{-s}sech^2 ax\, dx$, which is also an entire function of s. For $s \in \mathbf{C}$ define*

$$\phi_\Gamma(s;b,\chi) = \frac{T_\Gamma(s;b,\chi)}{\Gamma(s)} + \frac{a(G)}{8}C_G\chi(1)vol(\Gamma\backslash G)\cdot \sum_{j=0}^{(d-2)/2} m_j(b)\frac{K(s-j-1;b,a(G))}{s-j-1},$$

(3.8)

$$\phi_\Gamma^{(n)}(s;b,\chi) = \frac{T_\Gamma(s;b,\chi)}{\Gamma(s)} + \frac{\pi^{\frac12}}{4}C_G\chi(1)vol(\Gamma\backslash G)(b+n^2)^{-s+\frac12}\cdot \sum_{j=0}^{n} m_j(b)(b+n^2)^j\frac{\Gamma(s-j-\frac12)}{\Gamma(s-j)};$$

see (3.1), (3.2), (3.3), (3.4), and Table 1. Then for $\mathbf{R}e\ s > d/2$, $D_\Gamma(s;b,\chi) = \phi_\Gamma(s;b,\chi)$ for $G = SO_1(2n,1)$, $SP(n,1)$, $F_{4(-20)}$, or $SU(l,1)$ with l odd (up to local isomorphism). For $G = SO_1(2n+1,1)$ we have $D_\Gamma(s;b,\chi) = \phi_\Gamma^{(n)}(s;b,\chi)$ for $\mathbf{R}e\ s > d/2 = n + 1/2$.

$\phi_\Gamma(\ ;b,\chi)$, $\phi_\Gamma^{(n)}(\ ;b,\chi)$ provide therefore the meromorphic continuation of $D_\Gamma(\ ;b,\chi)$. We see, for example, that $\phi_\Gamma(\ ;b,\chi)$ is holomorphic except for possibly a simple pole at $s = p, p = 1, 2, \dots, d/2$, with residue $\frac14 C_G\chi(1)\mathrm{vol}(\Gamma\backslash G)m_{p-1}(b)$ (since $K(0;b,a) = 2/a$). In particular $m_{(d/2)-1}(b) \ne 0$ (see Table 1) $\Rightarrow d/2$ is indeed a simple pole. Also since $1/\Gamma(s)$ vanishes at $s = -1, -2, -3, \dots$, we can easily compute the value of $\phi_\Gamma(\ ;b,\chi)$ or of $\phi_\Gamma^{(n)}(\ ;b,\chi)$ at any negative integer. Justification can be given for the use of Fubini's theorem which leads to an expression of T_Γ in terms of the K-Bessel function K_ν:

(3.9)
$$T_\Gamma(s;b,\chi) = \frac{\pi^{-\frac12}}{\left[2\sqrt{b+\rho_0^2}\right]^{s-\frac12}} \sum_{\gamma\in C_\Gamma - \{1\}} \chi(\gamma)j(\gamma)^{-1}C(\gamma)t_\gamma^{s+\frac12}K_{-s+\frac12}\left(t_\gamma\sqrt{b+\rho_0^2}\right)$$

As an application of Theorem 3.7 we derive

THEOREM 3.10. *Let w denote the Weyl function*

$$x \to w(x) \overset{def}{=} \sum_{\lambda_j \le x} n_j(\chi) \quad \text{for} \quad x \ge 0. \tag{3.11}$$

Then w is finite-valued and $w(x) \sim (C_G/4)\chi(1)vol(\Gamma \setminus G)m_{(d/2)-1}(1)x^{d/2}$ as $x \to \infty$, with $m_{(d/2)-1}(1)$ replaced by $m_n(1) = 1$ in case $G = SO_1(2n+1,1)$; see (3.3), (3.4), and Table 1.

PROOF. Observe first that, after all, $D_\Gamma(s; b, \chi)$ is a Dirichlet series

$$\sum_{j=0}^{\infty} n_j(\chi) e^{-\beta_j s}$$

for $\beta_j \overset{\text{def}}{=} \log(b + \lambda_j)$. Since $\sum_{j=0}^{\infty} n_j(\chi)e^{-\beta_j \sigma}$ converges, say for $\sigma = (d/2) + 1$, that is $\lim_{j\to\infty} n_j(\chi)e^{-\beta_j\sigma} = 0$, given $x \ge 0$ $\exists$ an integer $N_x > 0$ $\ni$ $j \ge N_x \Rightarrow n_j(\chi)e^{-\beta_j\sigma} < e^{-[\log(b+x)]\sigma}$. Since each $n_j(\chi) \ge 1$ we can have $\lambda_j \le x$ only for $j < N_x$. $w(x)$ is therefore finite (as Gangolli has already observed). We choose $b = 1$ (i.e. $\beta_0 = 0$) and represent our Dirichlet series as a Laplace-Stieltjes transform:

$$\int_0^\infty e^{-st} d\alpha(t) = -n_0(\chi) + \sum_{j=0}^{\infty} n_j(\chi) e^{-\beta_j s}$$

$$\text{for } \alpha(t) \overset{\text{def}}{=} \sum_{\beta_j \le t} n_j(\chi) = w(e^t - 1),\ t \ge 0. \tag{3.12}$$

That is, by Theorem 3.7, $\int_0^\infty e^{-st}d\alpha(t)$ has the form

$$\int_0^\infty e^{-st} d\alpha(t) = \frac{g(s)}{(s-a)^p} + h(s) \quad \text{for } \mathbf{Re}\ s > a \tag{3.13}$$

where $a = d/2$, g and h are holomorphic for $\mathbf{Re}s \ge a$, $p \ne 0, -1, -2, -3, \ldots$, and $g(a) \ne 0$. Namely, $p = 1$ and for $A_\Gamma(\chi) = (a(G)/8)C_G\chi(1)\text{vol}(\Gamma \setminus G)$, $g(s) \overset{\text{(i)}}{=} A_\Gamma(\chi)m_{(d/2)-1}(1)K(s-d/2; 1, a(G))$ for $G \ne SO_1(2n+1,1)$ and $g(s) \overset{\text{(ii)}}{=} B_\Gamma(1+n^2)^{-s+n+1/2}\Gamma(s-n+1/2)/\Gamma(s-n)$ for $G = SO_1(2n+1,1)$ (using that $\Gamma(s-n-1/2)(s-(n+1/2)) = \Gamma(s-n+1/2)$), where $B_\Gamma(\chi) = (\pi^{1/2}/4)C_G\chi(1)$ $\cdot$ vol$(\Gamma \setminus G)$. Now given the format (3.13) one concludes by Delange [3], for example, that $\alpha(t) \sim (g(a)e^{at}t^{p-1})/\Gamma(p) = g(d/2)e^{(d/2)t}$ as $t \to \infty$. By (i), (ii), $g(d/2) = (1/4)C_G\chi(1)\text{vol}(\Gamma \setminus G)m_{(d/2)-1}(1)$ with $m_{(d/2)-1}(1)$ replaced by $m_n(1) = 1$ in case $G = SO_1(2n+1,1)$ (again using $K(0; 1, a) = 2/a$). By (3.12), $w(x) = \alpha(\log(x+1)) \sim g(d/2)(x+1)^{d/2}$, or $w(x) \sim g(d/2)x^{d/2}$ as desired, since $(x+1)^{d/2} \sim x^{d/2}$. □

As another application of Theorem 3.7 we give a factorization of the Selberg zeta function $Z_\Gamma(\cdot, \chi)$ attached to X_Γ; cf. [9, 17, 25]. Let Δ_Γ be the Laplacian of

section 1 and again let $A_\Gamma(\chi) \overset{\text{(iii)}}{=} (a(G)/8)C_G\chi(1)\text{vol}(\Gamma\backslash G)$. For $a, b > 0, c \in \mathbf{C}, n = 0, 1, 2, 3, \ldots$, define

$$I_n(b, a; c) = \int_{\mathbf{R}} t^{2n}(\operatorname{sech}^2 at)(b^2 + t^2)\left[\log(b^2 + t^2) - c\right] dt. \tag{3.14}$$

The integrals I_n are evaluated in [24], where it is shown, in particular, that $b \to I_n(b, a; c)$ admits a holomorphic continuation to the domain $\mathbf{C} - \{x + iy | y = 0, x \leq -a/2\pi\}$. The following result which we state for $\mathbf{Re}\ s > 2\rho_0$ therefore extends to a larger domain. For simplicity we omit the cases $G = SO_1(2n+1, 1), SU(l, 1)$ with l even.

THEOREM 3.15. *$\exists\ c(\chi) > 0$ such that*

$$Z_\Gamma(s, \chi) = c(\chi)\det\left[-\Delta_\Gamma + s(s - 2\rho_0)1\right]\left[e^{m_0(s(s-2\rho_0))I_0(s-\rho_0, a(G);1)} \cdot \prod_{j=1}^{\frac{d}{2}-1}\prod_{l=0}^{j} e^{\left(\frac{m_j(s(s-2\rho_0))}{j+1}\binom{j}{l}(s-\rho_0)^{2(j-l)}I_l(s-\rho_0, a(G);(1/(j+1)))\right)}\right]^{A_\Gamma(\chi)}; \tag{3.16}$$

see (3.4), definition (iii) preceding (3.14), and (3.14).

Here $\det(-\Delta_\Gamma + b1) \overset{\text{def}}{=} e^{-(\partial/\partial s)\phi_\Gamma(0;b,\chi)}$ (which is well-defined by Theorem 3.7; also see [14]) is the usual regularization of the determinant. Theorem 3.15 was obtained for $G = SL(2, \mathbf{R})$ by Sarnak [16], and Voros [18]; cf. [4, 6, 12] and other references in [23], where Theorem 3.15 is proved (in case $\chi = 1$). By working with the coefficients $m_j(b)$ in (3.4) rather than the coefficients of $P(r)$ (which we called Miatello coefficients in [22, 23]) we obtain simpler formulas in Theorems 3.7, 3.15.

For G locally isomorphic to $SU(l, 1)$ with l *even* (the cotangent case in (3.2)) we modify the definition of $\phi_\Gamma(s; b, \chi)$ in (3.8) by adding to it the following extra term: $(1/4)\chi(1)\text{vol}(\Gamma\backslash G)C_G\sum_{j=0}^{(d-2)/2=l-1} m_j(b)L_0(s - j; b, a(G) = \pi/2)$, where for $a > 0$

$$L_0(s; b, a) = \int_{\mathbf{R}} \frac{(x\ \operatorname{csch} ax)\operatorname{sech} ax\ dx}{(b + \rho_0^2 + x^2)^s}. \tag{3.17}$$

Then Theorem 3.7 holds. Here the integral in (3.17) converges uniformly on compact subsets of the plane (as one has an estimate $x\operatorname{csch} ax < Me^{-a|x|/2}$) and is thus an entire function of s. The format of equation (3.13) is thus unchanged and the proof of Theorem 3.10 is thus unchanged in the cotangent case.

In case Γ is allowed to have torsion there are central and elliptic contributions to the trace formula (in addition to the usual hyperbolic contribution). For the elliptic contributions (finite in number) we must show that the Mellin transform $s \to \int_0^\infty I_\gamma(h_t)t^{s-1}dt$ of the orbital integral $I_\gamma(h_t)$ of the fundamental solution h_t $(t > 0)$ of the heat equation on X is holomorphic, for elliptic $\gamma \in \Gamma$. Then as the central contributions are trivially handled, one obtains $w(x) \sim |\Gamma \cap \text{center of } G|(C_G/4)\chi(1)\text{vol}(\Gamma\backslash G)m_{(d/2)-1}(1)x^{d/2}$ in Theorem 3.10.

Throughout, our symmetric space is assumed to have rank equal 1, as in [5, 8]. In [7] G is assumed to possess a complex structure, and in [19] the rank is arbitrary. In all cases $\Gamma \setminus G$ is assumed to be compact.

Added in proof. In the four months since this paper was written the author has essentially worked out the extension of the main result Theorem 3.7 (and hence Theorem 3.10) to cover the case of $\Gamma \setminus G$ with finite volume.

REFERENCES

[1] Harish-Chandra, *Representations of semisimple Lie groups II*, Trans. Amer. Math. Soc. vol 76 (1954), 26-65.

[2] D. DeGeorge, *Length spectrum for compact locally symmetric spaces of strictly negative curvature*, Ann. École Norm. Sup. vol 10 (1977), 133-152.

[3] H. Delange, *Un théorèm sur les functions arithmétiques multiplicatives et ses applications*, Ann. Sci. École Norm. Sup. (3) vol 78 (1961), 1-29.

[4] E. D'Hoker, D. Phong, *On determinants of Laplacians on Riemann surfaces*, Comm. Math. Physics vol 104 (1986), 537-545.

[5] T. Eaton, *Asymptotic behavior of spectra of compact quotients of symmetric spaces of rank one of the noncompact type*, Ph.D. Thesis, Univ. of Washington, 1973.

[6] D. Fried, *Analytic torsion and closed geodesics on hyperbolic manifolds*, Invent. Math. vol 84 (1986), 523-540.

[7] R. Gangolli, *Asymptotic behavior of spectra of compact quotients of certain symmetric spaces*, Acta Math. vol 121 (1968), 151-192.

[8] R. Gangolli, *On the length spectrum of certain compact manifolds of negative curvature*, J. Diff. Geometry vol 12 (1977), 403-424.

[9] R. Gangolli, *Zeta functions of Selberg's type for compact space forms of symmetric spaces of rank one*, Ill. J. Math. vol 21 (1977), 1-42.

[10] R. Gangolli, G. Warner, *On Selberg's trace formula*, J. Math. Soc. Japan vol 27 (1975), 328-343.

[11] I. M. Gelfand, *Automorphic functions and the theory of representations*, In: Proceedings of the Internat. Congress of Mathematicians, Stockholm, 1962, Inst. Mittag-Leffler, Djursholm (1963), 74-85.

[12] S. Koyama, *Determinant expression of Selberg zeta functions I*, TAMS vol 324 (1991), 149-168.

[13] R. Miatello, *On the Plancherel measure for linear Lie groups of rank one*, Manuscripta Math. vol 29 (1979), 249-276..

[14] S. Minakshisundaram, A. Pleijel, *Some properties of the eigenfunctions of the Laplace operator on Riemannian manifolds*, Canadian J. Math. vol 1 (1949), 242-256.

[15] B. Randol, *On the analytic continuation of the Minakshisundaram-Pleijel zeta function for compact Riemann surfaces*, Trans. Amer. Math. Soc. vol 201 (1975), 241-246.

[16] P. Sarnak, *Determinants of Laplacians*, Comm. Math. Physics vol 110 (1987), 113-120.

[17] A. Selberg, *Harmonic analysis on discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. vol 20 (1956), 47-87.

[18] A. Voros, *Spectral functions and the Selberg zeta function*, Comm. Math. Physics vol 110 (1987), 439-465.

[19] N. Wallach, *On an asymptotic formula of Gelfand and Gangolli for the spectrum of* $\Gamma \setminus G$., J. Diff. Geometry vol 11 (1976), 91-101.

[20] N. Wallach, *On the Selberg trace formula in the case of compact quotient*, Bull. AMS vol 82 (1976), 171-195.

[21] F. Williams, *Formula for the class 1 spectrum and the meromorphic continuation of Minakshisundaram-Pleijel series*, preprint, 1989.

[22] F. Williams, *Some zeta functions attached to* $\Gamma \setminus G/K$, In: *New Developments in Lie Theory and Their Applications, (J. Tirao, N. Wallach, eds.)* Birkhäuser Progress in Math. Ser. vol 105.

[23] F. Williams, A factorization of the Selberg zeta function attached to a rank 1 space form, Manuscripta Math. vol 77 (1992), 17-39.
[24] F. Williams, On certain definite integrals which arise in automorphic Lie theory, J. Physics A, vol 26 (1993), 3515-3526.
[25] F. Williams, *Lectures on the spectrum of* $L^2(\Gamma \backslash G)$, Pitman Research Notes in Math. **242**, Longman House Pub., 1990.

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Representation Theory and Harmonic Analysis.
A Conference in Honor of Ray A. Kunze
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and Paul J. Sally, Jr., Editors

This volume is an outgrowth of the special session on representation theory and harmonic analysis held in honor of Ray Kunze at the 889th meeting of the American Mathematical Society on January 12–15, 1994 (Cincinnati). The mathematicians who gathered together for the special session included many of Kunze's former Ph.D. students, research collaborators, and others whose work has been influenced either directly or indirectly through their associations with Kunze. As the session progressed, it became clear that the papers being presented spanned a wide spectrum of group representations and harmonic analysis and their applications that would be of interest to a more extensive audience.

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